From classical to the quantum strings

Miroslav Pardy

Department of Physical electronics and
the laboratory of plasma physics
Masaryk University
Kotlářská 2, 611 37 Brno, Czech Republic
email:pamir@physics.muni.cz

December 3, 2021

Abstract

We consider some problems concerning the classical string and the quantum strings which can have the deep physical meaning. We show that there is the asymmetry of the action and reaction in the string motion for the string with the left end fixed and the right end being in periodic motion. We, show that the law of the action-reaction symmetry is violated during the string motion. We derive also the quantum internal motion of this system.

The quantization of the string with the interstitial massive defect is performed. It is not excluded that the solution is related to the Mössbauer effect, being recoilless nuclear resonance fluorescence, which is the resonant and recoilfree emission and absorption of gamma radiation by atomic nuclei bound in a solid.

The classical motion of uniformly accelerated string and its relation the Bell paradox is considered. In this case the string with the length l, is accelerated in such way the left end and the right end of which is non-relativistically and then relativistically accelerated by the constant acceleration a. We discuss the relation of this accelerated string to the Bell spaceship paradox involving the Lorentz contraction. It is evident that the acceleration of the string can be caused by gravity and we show that such acceleration causes the different internal motion of the string.

Gravity can be described by metrics in the Einstein theory of gravity and by string medium in the Newton model of gravity. We show, that in case of the string model of gravity the motion of planets and Moon are oscillating along the classical trajectories

In the string model of hadrons the quarks are treated to be tied together by a gluon tube which can be approximated by the tube of vanishing width, or by string. We apply the delta-function form of force to the left side of the rod and calculate the propagation of the pulse in the system.

PREFACE

We consider in this elaborate some problems concerning the classical string and the quantum strings which can have the deep physical meaning. The **first part** concerns the asymmetry of the action and reaction in the string motion. We consider the string, the left end of which is fixed and the right end of this string is in periodic motion. We, show that the law of the action-reaction symmetry is broken during the string motion.

According to the third Newton's law of motion, all forces occur in pairs in such a way that if one object exerts a force on another object, then the second object exerts an equal and opposite reaction force on the first. In other words: "To every action there is always opposed an equal reaction: or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts" (Halliday et al., 1966). We will easily see in the string mediation of force that the third Newton law of action and reaction is broken.

In the **second part** of our composition we consider the classical motion of uniformly accelerated string and its relation the Bell paradox. We consider the string with the length l, the left end and the right end of which is non-relativistically and then relativistically accelerated by the constant acceleration a. We calculate the motion of the string with no intercalation of the Fitzgerald contraction of the string. We consider also the Bell spaceship paradox. The Bell paradox and our problem is in the relation with the Lorentz contraction in the Cherenkov effect (Pardy, 1997) realized by the carbon dumbbell moving in the LHC or ILC (Pardy, 2008). The Lorentz contraction and Langevin twin paradox (Pardy, 1969) is interpreted as the Fock measurement procedure (Fock, 1964;).

In the **third part** of our physical work we discuss the free fall of the string in gravity, which is the analogue of the Galileo experiment performed in Pisa.

It is well known that Galileo performed experiment - later the famous experiment - with the result that the every falling body is falling with a uniform acceleration, the resistance of the medium being through which it was falling remained negligible. He also derived the correct kinematic law for the distance traveled during a uniform acceleration starting from rest, namely, that it is proportional to the square of the elapsed time. Galileo expressed the time-squared law using geometrical constructions and mathematically precise words. The solutions are not identical with the string accelerated kinetically by acceleration a. So, we distinguish between non-inertial field and the gravity field and we discuss the principle of equivalence. In conclusion we suggest to drop the charged objects from the very high tower Burj Khalifa in order to say crucial words on the principle of equivalence.

In the **forth part** of our document we consider the string model of gravity leading to the zitterbewegung of planets and Moon in or planetary system.

The string model of gravitational force was proposed by author 40 years ago (Pardy, 1980; 1996). In this model the string forms the mediation of the gravitational interaction between two gravitating bodies. It reproduces the Newtonian results in the first-order approximation and it predicts in the higher-oder approximations the existence of oscillations of the massive bodies interacting by the string. In case of the Moon it can be easily verified by the NASA laser measurements.

Instead of resolution of this problem Newton suggested the phenomenological theory of the gravitational force, where there exists no answer concerning the dynamics, or, the mechanism of action-at-a-distance. Newton himself was award that it necessary exists some mediation of interaction between two bodies at the different points in space because he has written (Newton. 1966): *It is inconceivable, that inanimate brute matter, should without the mediation of something else which is not material, operate upon and effect other matter without mutual contact .. .* In other words, the crucial notion in the Newton speculation is the mediation between two bodies. Now, we can say that Newton considered the string model of gravity because string model of gravity is the natural result of Newton scientific intuition.

We will consider the string, the left end of which is fixed at the beginning of the coordinate system and mass m is fixed on the right end of the string. The motion of the system string and the body with mass m is the fundamental problem of the equations of the mathematical physics in case that the tension is linearly dependent on elongation (Tikhonov et al., 1977). We will show that it is possible to represent the Newton gravitational law by the string with the nonlinear tension in the string. Because of the strong nonlinearity of the problem the motion of the string and the body can be solved only approximately. In the following text, we will give the approximative solution of the classical two-body problem and then we obtain the string solution of this problem.

In the **fifth part** of our investigations we deal with the pulse propagation in the string with the massive ends. We consider the elastic rod of a large mass M, the left end of which is fixed to a body of mass $m << M$ and the second body of mass m is fixed to the right end of the rod. The force of the delta-function form is applied to the left side of the rod. We find the propagation of the pulse in the system. Our problem represents the missing problem in the textbooks on mechanics. The relation of our theory to the quark-string model of mesons is evident.

In the string model of hadrons the quarks are treated to be tied together by a gluon tube which can be approximated by the tube of vanishing width, or by string (Nambu, 1974). Then, dynamics of hadrons can be approximated by the Nambu-Goto action for the relativistic string. The merit of the string model is the natural explanation of the quark confinement and the dynamics of the system. On the other hand, there are some mathematical problems when the quark masses are different from zero. So, there are many trials to give the final words to the problem of hadron dynamics and hadron masses. It is well known that some models of string theory involves also the so called extra-dimensions. However, it is well known that this theory is at present time not predictable. The goal of theoretical physics is predictions based on the mathematical knowledge. On the other hand, the goal of the experimental physic is the confirmation of theory with appropriate experimental simplicity. We use here approach to the string dynamics which consists in experimental situation in laser physics that the pulse is applied to the left side of a string and then we determine the string dynamics. We do not consider in this simplification the rotating string.

In the **sixh part** of our booklet we consider the string motion under the periodic boundary conditions. We consider the string, the left end of which is fixed and the right end of the string is in periodic motion. We derive the quantum internal motion of this system. According to Nielsen and Olesen (1973) there is parallelism between the Higgs model of broken gauge invariance and the Landau-Ginzburg superconductivity theory on the one hand and and the dual string model and at the same time between Abrikosov liquid lines in superconductors II. So dual string is mathematical realization of magnetic liquid tube in equilibrium against the pressure of the surrounding charged superliquid.

Only strings with no ends were considered (Nambu, 1974). The internal quantum motion of strings is not considered by the authors. We consider here the string, the left end of which is fixed and the right end of the string is in periodic motion. We derive the quantum internal motion of this system. We use the so called the oscillator quantization of string.

In the **seven part** of our composition we he string motion under the periodic local force. The string is under the local periodic force $F = \varrho A \sin \omega t$ at point $0 < c < l$. With regard to the fact that we consider the additional periodical force at point $0 < c < l$, we must reformulate the standard string problem of mathematical physics. The motion of the string is u_1 in the interval $0 < x < c$ and u_2 in the interval $c < x < l$. Then we use the oscillator method of quantization for obtaining the quantum motion of such system.

In the **eight part** of our script we perform the quantization of the string motion with the interstitial massive defect. We will consider the string, the left end of which is fixed at the beginning of the coordinate system, the right end is fixed at point l and mass m is fixed between the ends of the string. We determine the classical and the quantum vibration of such system. The proposed model can be also related in the modified form to the problem of the Mössbauer effect, being recoilless nuclear resonance fluorescence, which is the resonant and recoilfree emission and absorption of gamma radiation by atomic nuclei bound in a solid.

In the last part - **Appendix**, we give some historical information on the action at a distance and some remarks on the meson model as the string model with two quarks.

1 The asymmetry of the action and reaction

1.1 Introduction

We consider the string, the left end of which is fixed and the right end of this string is in periodic motion. We, show that the law of the action-reaction symmetry is broken during the string motion.

According to the third Newton's law of motion, all forces occur in pairs in such a way that if one object exerts a force on another object, then the second object exerts an equal and opposite reaction force on the first. In other words: "To every action there is always opposed an equal reaction: or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts" (Halliday et al., 1966). We will easily to see in the string case that the third Newton law of action and reaction is broken.

1.2 The classical derivation of the string motion

The differential equation of motion of string elements can be derived by the following way (Tikhonov et al., 1977). We suppose that the force acting on the element dx of the string is given by the Hook law:

$$T(x,t) = ES\left(\frac{\partial u}{\partial x}\right), \tag{1}$$

where E is the modulus of elasticity, S is the cross section of the string. We easily derive that

$$T(x+dx) - T(x) = ESu_{xx}dx. \tag{2}$$

The mass dm of the element dx is $\varrho ESdx$, where $\varrho = const$ is the mass density of the string matter and the dynamical equilibrium gives

$$\varrho Sdxu_{tt} = ESu_{xx}dx. \tag{3}$$

So, we get

$$\frac{1}{c^2}u_{tt} - u_{xx} = 0; \quad c = \left(\frac{E}{\varrho}\right)^{1/2}. \tag{4}$$

Now, let us consider the following problem of the mathematical physics. The right end of the string is in the periodic motion $u(l,t) = A\sin(\omega t)$. So we solve the mathematical problem:

$$u_{tt} = c^2 u_{xx} \tag{5}$$

with the initial conditions

$$u(x,0) = 0; \quad u_t(x,0) = 0 \tag{6}$$

and with the boundary conditions

$$u(0,t) = 0; \quad u(l,t) = A\sin(\omega t). \tag{7}$$

The equation (5) with the initial and boundary conditions (6) and (7) represents one of the standard problems of the mathematical physics and can be easily solved using the the standard methods . The solution is elementary (Lebedev et al., 1955) and it is the integral part of equations of mathematical physics (Tikhonov et al., 1977):

$$u(x,t) = \frac{Ac}{ES\omega}\left[\frac{\sin\frac{\omega x}{c}}{\frac{\cos\omega l}{c}}\right]\sin(\omega t). \tag{8}$$

So, we see that the string motion is a such that at every point $X \in (0,l)$ there is an oscillator with an amplitude

$$\mathcal{A} = \frac{Ac}{ES\omega}\left[\frac{\sin\frac{\omega x}{c}}{\frac{\cos\omega l}{c}}\right]; \quad x \in (0,l). \tag{9}$$

1.3 Action and reaction inside the string

The force in the form of the local tension inside the string (1) can be easily computed in the left boundary and in the right boundary. So,

$$T(0,t) = ES\left(\frac{\partial u}{\partial x}\right)_{x=0} = A\left[\frac{1}{\frac{\cos\omega l}{c}}\right]\sin(\omega t) \tag{10}$$

and

$$T(l,t) = ES\left(\frac{\partial u}{\partial x}\right)_{x=l} = A\left[\frac{\cos\omega(l/c)}{\frac{\cos\omega l}{c}}\right]\sin(\omega t). \tag{11}$$

So, we see,

$$T(0,t) \neq T(l,t), \tag{12}$$

Q.E.D.

1.4 Discussion

The string theory can be extended to the quark-quark interaction by the string potential, defined as the quark mass correction to the string potential, which was performed by Lambiase and Nesterenko (1996). It is evident that the relation (12) is valid in such situation. The calculation of the interquark potential generated by a string with massive ends was performed by Nesterenko and Pirozhenko (1997) and others with the validation of the theorem (12). The propagation of a pulse in the real strings and rods which can be applied to the two-quark system of pion and so on, was calculated by author (Pardy, 2005), and the relation (12) is valid. Also, in case of the string model of gravity, leading to the Zitterbewegung of planets and Moon, (Pardy, 2020), the theorem (12) is valid. So, it is not excluded that our approach can be extended to generate the new way of the string theory of matter and space-time.

2 The uniformly accelerated string and the Bell paradox

2.1 Introduction

We consider the string with the length l, the left end and the right end of which is non-relativistically and then relativistically accelerated by the constant acceleration a. We calculate the motion of the string with no intercalation of the Fitzgerald contraction of the string. We consider also the Bell spaceship paradox. The Bell paradox and our problem is in the relation with the Lorentz contraction in the Cherenkov effect (Pardy, 1997) realized by the carbon dumbbell moving in the LHC or ILC (Pardy, 2008). The Lorentz contraction and Langevin twin paradox (Pardy, 1969) is interpreted as the Fock measurement procedure (Fock, 1964;).

2.2 The non-relativistic acceleration of the string

The differential equation of motion of string element was be derived in part 1. The procedure was performed evidently in order to get the wave equation. Our problem is described by the wave equation (Koshlyakov, et al., 1962).

$$u_{tt} = c^2 u_{xx} + g(x,t); \quad c = \left(\frac{E}{\varrho}\right)^{1/2}, \tag{1}$$

where $g(x,t) = p(x,t)/\varrho$, $p(x,t)$ being a force and the boundary conditions are

$$u(x=0) = \kappa_1(t) = \frac{1}{2}at^2, \tag{2}$$

$$u(x=l) = \kappa_2(t) = \frac{1}{2}at^2 + l = \kappa_1(t) + l. \tag{3}$$

The initial conditions are

$$u(t=0) = f(x); \quad u_t(t=0) = F(x). \tag{4}$$

The problem cannot be solved by the standard Fourier method because the boundary conditions (2)-(3) are not homogeneous. So, we introduce the auxiliary function (Koshlyakov et al., 1962)

$$w(x,t) = \kappa_1(t) + [\kappa_2(t) - \kappa_1(t)]\frac{x}{l} \tag{5}$$

with the boundary conditions

$$w(x=0) = \kappa_1(t); \quad w(x=l) = \kappa_1(t) + l \tag{6}$$

and we take the solution in the form:

$$u(x,t) = v(x,t) + w(x,t). \tag{7}$$

with the boundary conditions

$$v(x=0) = 0; \quad v(x=l) = 0 \tag{8}$$

and with the initial conditions

$$v(t=0) = f_1(x); \quad v_t(t=0) = F_2(x). \tag{9}$$

After insertion of $u = v + w$ into ew (1), we get the following equation for v and w:

$$v_{tt} = c^2 v_{xx} + g(x,t) + c^2 w_{xx} - u_{tt}. \tag{10}$$

Then, if we use the definition of w by eq. (5), we get equation for v in the form:

$$v_{tt} = c^2 v_{xx} + g_1(x,t), \tag{11}$$

where

$$g_1(x,t) = g(x,t) - \kappa_1''(t) - [\kappa_2''(t) - \kappa_1''(t)]\frac{x}{l} \tag{12}$$

So, we see, that the last algebraic procedures lead to new system of equations. Namely:

$$v_{tt} = c^2 v_{xx} + g_1(x,t) \tag{13}$$

with

$$v(x=0) = 0; \quad v(x=l) = 0 \tag{14}$$

and

$$v(t=0) = f_1(x); \quad v_t(t=0) = F_2(x) \tag{15}$$

It is easy to show that $g_1(x,t) = g - a$ and the system of equation to be solved is as follows:

$$v_{tt} = c^2 v_{xx} + g - a \tag{16}$$

with

$$v(x=0)=0; \quad v(x=l)=0 \tag{17}$$

and

$$v(t=0)=f_1(x)=0; \quad v_t(t=0)=F_1(x)=0 \tag{18}$$

The solution of the system is well known (Koshlyakov et al., 1962) and so we write the final form:

$$v(x,t)=\sum_{k=1}^{\infty} T_k \sin\left(\frac{k\pi x}{l}\right), \tag{19}$$

where

$$T_k(t)=\frac{2}{l\omega_k}\int_0^t d\tau \int_0^l G(\xi,\tau)\sin\omega_k(t-\tau)\sin\left(\frac{k\pi\xi}{l}\right)d\xi, \tag{20}$$

where

$$\omega_k=\frac{k\pi c}{l}; \quad G(\xi,\tau)=g-a. \tag{21}$$

So, $u=v+w=v+\frac{1}{2}at^2+x$.

2.3 The relativistic acceleration of the string

Let us consider, at first, the relativistic uniformly accelerated motion of a particle, i.e. the rectilinear motion for which the acceleration w in the proper reference frame (at each instant of time) remains constant.

In the reference frame where the particle velocity is $v=0$, the components of the four-acceleration is $a'=(0,a/c^2,0,0)$, where a is the ordinary three-dimensional acceleration directed along the x axis and c is here the velocity of light. The relativistically invariant condition for uniform acceleration is the constancy of the four-scalar which coincides with a^2 in the proper reference frame (Landau, et al., 1987):

$$-\frac{a^2}{c^4}=a^i a_i \equiv const \tag{22}$$

In the "fixed" frame, with respect to which the motion is observed, writing out the expression for $a^i a_i$ gives the equation

$$\frac{d}{dt}\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}=a, \tag{23}$$

or,

$$\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}=at+const, \tag{24}$$

Setting $v = 0$ for $t = 0$, we find that $const = 0$, so that

$$v = \frac{vt}{\sqrt{1 - \frac{a^2t^2}{c^2}}} = at + const, \tag{25}$$

Integrating once more and setting $x = 0$ for $t = 0$, we find:

$$x = \frac{c^2}{a}\left(\sqrt{1 + \frac{a^2t^2}{c^2}} - 1\right) \tag{26}$$

For $at \ll c$, these formulas go over the classical expressions $v = at, x = at^2/2$. For $at \to \infty$, the velocity tends toward the constant value c.

The proper time of a uniformly accelerated particle is given by the integral

$$\int_{0,}^{t} \sqrt{1 - \frac{v^2}{c^2}} = \frac{c}{a}\sinh^{-1}\frac{at}{c} \tag{27}$$

As $t \to \infty$ it increases much more slowly than t, according to the law $c/a \log(2at/c)$ (Landau, et al., 1987).

The solution is the same as in case of the non-relativistic motion, only with the replacing eqs. (2-3) by the equations with the relativistic motion, or with the boundary conditions

$$u(x = 0) = \kappa_1(t) = \frac{c^2}{a}\left(\sqrt{1 + \frac{a^2t^2}{c^2}} - 1\right) \tag{28}$$

$$u(x = l) = \kappa_2(t) = \frac{c^2}{a}\left(\sqrt{1 + \frac{a^2t^2}{c^2}} - 1\right) + l = \kappa_1(t) + l \tag{29}$$

and with the initial conditions

$$u(t = 0) = f(x); \quad u_t(t = 0) = F(x). \tag{30}$$

The final solution of the system is the analogue of eqs. (19–21) (Koshlyakov et al., 1962). So, we write the final form:

$$v(x,t) = \sum_{k=1}^{\infty} T_k \sin\left(\frac{k\pi x}{l}\right), \tag{31}$$

where

$$T_k(t) = \frac{2}{l\omega_k}\int_0^t d\tau \int_0^l \Gamma(\xi,\tau)\sin\omega_k(t-\tau)\sin\left(\frac{k\pi\xi}{l}\right)d\xi, \tag{32}$$

with

$$\omega_k = \frac{k\pi(E/\varrho)^{1/2}}{l} \tag{33}$$

and the function $\Gamma(\xi, \tau)$ is after some calculation given by the formula:

$$\Gamma(\xi, \tau) = g - a\left(1 + \frac{a^2\tau^2}{c^2}\right)^{-3/2}. \tag{34}$$

2.4 The Bell spaceship and the accelerated string

The Bell thought experiment can be considered evidently as the analogue of our situation with the accelerated string. In the Bell version of the thought experiment (Bell, 1993), three spaceships A, B and C are initially at rest in a common inertial reference frame, B and C being equidistant to A. Then, a signal is sent from A to B and C int its simultaneously detected by B and C, causing B and C starting to accelerate in the vertical direction with identical acceleration, while A stays at rest in its original reference frame.

Then, from the observer in A, - B, C, will have at every moment the same velocity, and so remain displaced one from the other by a fixed distance. Let fragile thread is tied initially between B and C . Then, as the rockets speed up, it will become too short, because of the Fitzgerald length contraction, and must finally break. It must break when, at a sufficiently high velocity, the artificial prevention of the natural contraction imposes intolerable stress. According to Bell, there was "clear consensus" which asserted, incorrectly, that the string would not break.

However, Petkov (2009) and Franklin (2009) interpret this paradox differently. They agreed with the result that the string will break **due to unequal accelerations** in the rocket frames, which causes the rest length between them to increase. However, they denied the idea that those stresses are caused by length contraction. This is because length contraction has no "physical reality", but is the result of a Lorentz relativistic transformation of space and time which by itself can never cause any stress at all. Thus, the occurrence of such stresses in all reference frames is supposed to be the effect of relativistic acceleration alone.

So, we see that the verbal argumentation by Bell and Petkov and others is not unambiguous. In our case we use the mathematically rigorous argumentation and it means that what is calculated by us is the realistic behavior of accelerated string.

2.5 Discussion

We have seen how to calculate the internal motion of the uniformly accelerated non-relativistic and the relativistic string of the length l. The initial length of the string is in our calculation constant and every modification of length by the Fitzgerald contraction is not acceptable in the theory of the algorithm of calculation. So, the question arises if our mathematical approach is in contradiction with the Bell verbal argumentation.

Let us remark that the Bell problem can be related to the author investigation on Lorentz contraction in the Čerenkov effect (Pardy, 1997) due to the carbon dumbbell moving in the LHC or ILC (Pardy, 2008). The Lorentz contraction is not interpreted in the sense of the Fitzgerald contraction bat in the sense of the measurement procedure by Fock (1964). Such interpretation, where the relativity is interpreted in the sense of the measurement procedure of Fock was also used in the article on the Langevin twin paradox, or, in other words, on the clock paradox (Pardy, 1969).

b

3 The Free Fall of the String in gravity

3.1 Introduction

We consider the motion of string in free fall in gravity. The solutions are not identical with the string accelerated kinetically by acceleration a. So, we distinguish between non-inertial field and the gravity field and we discuss the principle of equivalence. In conclusion we suggest to drop the charged objects from the very high tower Burj Khalifa in order to say crucial words on the principle of equivalence.

It is well known that Galileo performed experiment in Pisa - later the famous experiment - with the result that the every falling body is falling with a uniform acceleration, the resistance of the medium being through which it was falling remained negligible. He also derived the correct kinematic law for the distance traveled during a uniform acceleration starting from rest, namely, that it is proportional to the square of the elapsed time. Galileo expressed the time-squared law using geometrical constructions and mathematically precise words.

We here repeat the Galileo experiment in the virtual mathematical form. Namely, with the string. We discuss the motion of the string with accelerated boundary conditions, and by gravity, and we discover substantial differences leading to the adequate philosophy of the principle of equivalence.

It is possible to show that the uniformly accelerated string of the length l, where the left end and the right end is accelerated by constant acceleration a, forms the mathematical problem described by the wave equation (Koshlyakov, et al., 1962)

$$u_{tt} = c^2 u_{xx} + g(x,t), \tag{1}$$

where $g(x,t) = p(x,t)/\varrho$ and $p(x,t)$ is the external force, and the boundary conditions are

$$u(x=0) = \frac{1}{2}at^2; \quad u(x=l) = \frac{1}{2}at^2 + l \tag{2}$$

and the initial conditions being

$$u(t=0) = f(x); \quad u_t(t=0) = F(x). \tag{3}$$

The solution of the system is well known , $u = v + w$ (Koshlyakov et al., 1962), where w is the solution of the homogeneous equation (1) (g=0) with the initial conditions

$$w(x = 0) = 0; \quad w_t(l = 0) = 0 \tag{4}$$

and with the boundary conditions

$$w(l = 0) = f(x); \quad w_t(l = 0) = F(x). \tag{5}$$

The solution v is derived in the final form (Koshlyakov et al., 1962):

$$v(x,t) = \sum_{k=1}^{\infty} T_k \sin\left(\frac{k\pi x}{l}\right), \tag{6}$$

where

$$T_k(t) = \frac{2}{l\omega_k}\int_0^t d\tau \int_0^l G(\xi,\tau)\sin\omega_k(t-\tau)\sin\left(\frac{k\pi\xi}{l}\right)d\xi, \tag{7}$$

with

$$\omega_k = \frac{k\pi c}{l}; \quad G(\xi,\tau) = g - a. \tag{8}$$

3.2 The Free fall of the string in gravity

Now, let us consider the string with length l, the upper end is hanged in the gravity with the acceleration g and the second end is free at time $t = 0$. So the mathematical formulation of the problem is as follows (Koshlyakov, et al., 1962):

$$u_{tt} = c^2 u_{xx} + g \tag{9}$$

with

$$u(x = 0) = 0; \quad u_x(x = l) = 0 \tag{10}$$

$$u(t = 0) = 0; \quad u_t(t = 0) = 0. \tag{11}$$

Putting $u = v + w$, we get for w the obligate system of equations:

$$w_{tt} = c^2 w_{xx} \tag{12}$$

with the boundary conditions

$$w(x = 0) = 0; \quad w_x(x = l) = 0 \tag{13}$$

and the initial conditions

$$w(t = 0) = -v(t = 0); \quad w_t(t = 0) = -v_t(t = 0). \tag{14}$$

It is possible to show (Koshlyakov, et al., 1962) that

$$v = \frac{gx(2l - x)}{2c^2}. \tag{15}$$

So, we can write

$$f(x) = \frac{gx(x - 2l)}{2c^2}; \quad F(x) = 0. \tag{16}$$

Then, by the standard method of integration, we get

$$u(x,t) = \frac{gx(2l - x)}{2c^2} -$$

$$\frac{16gl^2}{\pi^3 c^2} \sum_{k=1}^{\infty} \frac{1}{(2k+1)^3} \cos\left(\frac{(2k+1)\pi at}{2l}\right) \sin\left(\frac{(2k+1)\pi x}{2l}\right) \tag{17}$$

and

$$u(x = l) = \frac{gl^2}{2c^2} - \frac{16gl^2}{\pi^3 c^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)^3} \cos\left(\frac{(2k+1)\pi at}{2l}\right). \tag{18}$$

The maximal quantity u_{max} is at point $t = 2l/c$ and so we get

$$u_{max} = \frac{gl^2}{2c^2} + \frac{16gl^2}{\pi^3 c^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)^3}. \tag{19}$$

With regard to the mathematical formula

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)^3} = \frac{\pi^3}{32}, \tag{20}$$

we get

$$u_{max} = \frac{gl^2}{c^2}. \tag{21}$$

So, the length of the string (rod) is in the interval $(l, l + \frac{gl^2}{c^2})$.

3.3 The string in the Einstein theory

The Einstein Gravity is based on the Einstein-Hilbert field equations (EHFE). They are the space-time geometry equations for the determining of the metric tensor of space-time for a given arrangement of stressenergy in the space-time. They are the non-linear partial differential equations and the solutions of the EHFE are the components of the metric tensor.

The inertial trajectories of particles are geodesics in the resulting geometry calculated using the geodesic equation.

The EHFE obeying local energy-momentum conservation, they reduce to the Newton law of gravitation where the gravitational field is weak and velocities are much less than the speed of light.

There is the simple derivation of the EHFE given by Fock (1964). The similar derivation was performed by Chandrasekhar (1972), Kenyon (1996), Landau et al. (1987), Rindler (2003) and others. Source theory derivation of Einstein equations was performed by Schwinger (1970).

It is well known that the gravity mass M_G of some body is equal to the its inertial mass M_I, where gravity mass is a measure of a massive body to create the gravity field (or, gravity force) and the inertial mass of a massive body is a measure of the ability of the resistance of the body when it is accelerated. At present time we know, that if components of elementary particles have the same gravity and inertial masses, the body composed with such elementary particles has the identical gravity and inertial mass. There is no need to perform experimental verification. So, particle physics brilliantly confirms the identity of the inertial and gravity masses.

According to the Newton theory, the gravity potential is given by the equation

$$U(r) = -\kappa\frac{M}{r}, \tag{22}$$

where r is a distance from the center of mass of a body, κ is the gravitational constant and its numerical value is in SI units $6.67430(15)10^{-11}m^3.kg^{-1}.s^{-2}$ (CODATA, 2018).

The potential U is as it is well known the solution of the Poisson equation:

$$\Delta U(r) = -4\pi\kappa\varrho, \tag{23}$$

where ϱ is the density of the distributed masses.

The problem is, what is the geometrical formulation of gravity equation (23) following from the space-time element ds, which has the specific form in case of the special theory of relativity.

Let us postulate that the motion of a body moving in the g-field is determined by the variational principle

$$\delta\int ds = 0. \tag{24}$$

In order to get the Newton equation of motion, we are forced to perform the following identity:

$$g_{00} = c^2 - 2U = -4\pi\kappa\varrho. \tag{25}$$

The second mathematical requirement, which has also the physical meaning is the covariance of the derived equation. It means that the necessary mathematical operation are the following replacing of original symbols:

$$U \to g_{\mu\nu} \tag{26}$$

with

$$\Delta U \to Tensor\ equation \tag{27}$$

and

$$\varrho \to T_{\mu\nu}, \tag{28}$$

where $T_{\mu\nu}$ is the tensor of energy and momentum.

In order to get the tensor generalization of eq. (23) it is necessary to construct new tensor $R_{\mu\nu}$, which is linear combination of the more complicated tensor $R_{\alpha\beta,\mu\nu}$, or

$$R_{\mu\nu} = g^{\alpha\beta} R_{\mu\alpha,\beta\nu} \tag{29}$$

and the scalar quantity R, which is defined by equation

$$R = g^{\lambda\mu} R_{\lambda\mu} \tag{30}$$

and construct the combination tensor $G_{\lambda\mu}$ of the form

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{31}$$

which has the mathematical property, that the covariant divergence of this tensor is zero, or,

$$\nabla^{\lambda} G_{\lambda\mu} = 0. \tag{32}$$

.

With regard to the fact that also the energy-momentum tensor $T_{\mu\nu}$ has the zero divergence, we can identify eq. (31) with the tensor $T_{\mu\nu}$, or

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi\kappa}{c^2} T_{\mu\nu}, \tag{33}$$

where the appeared constant in the last equation is introduce to get the classical limit of the equation.

The approximate solution of the last equation is as follows

$$ds^2 = (c^2 - 2U)dt^2 - \left(1 + \frac{2U}{c^2}(dx^2 + dy^2 + dz^2)\right). \tag{34}$$

The space-time element (34) is able to explain the shift of the frequency of light in gravitational field and the deflection of light in the gravitational field of massive body with mass M.

So, we have seen that the basic mathematical form of the Einstein general relativity is the Riemann manifold specified by the metric with the physical meaning. The crucial principle is the equality of the inertial and gravitational masses.

While the derivation of the EHFE is elementary, Feynman wrote that the derivation of EHFE by Einstein is difficult to understand. Namely: *Einstein himself, of course, arrived at the same Lagrangian but without the help of a developed field theory, and I must admit that I have no idea how he guessed the final result. We have had troubles enough arriving at the theory - but I feel as though he had done it while swimming underwater, blindfolded, and with his hands tied behind his back!* (Feynman et al., 1995).

Now the question arises, what is the equation of motion of the string in a gravitational field. The general solution is beyond of the possibility of mathematical physics and the specific case is of no easy solution. Namely, the force acting on the point moving is the homogeneous gravitational field was calculated in the 3-form as follows (Landau, et al., 1988):

$$\mathbf{f} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}\left\{\mathbf{grad}\ln\sqrt{h} + \sqrt{h}\left[\frac{\mathbf{v}}{c}\mathbf{rot}\ \mathbf{g}\right]\right\}, \tag{35}$$

with (Landau, et al., 1988).

$$h = 1 + \frac{2\varphi}{c^2}, \tag{36}$$

where φ s gravitational potential generating the acceleration $\mathbf{g}$.

So, we see that if we perform the application of the last formula on the string motion, the problem is beyond of the problems of the university physics. So, we have decided for the classical solution in the framework of the equations of mathematical physics.

3.4 Discussion

We have seen how to calculate the internal motion of the uniformly accelerated non-relativistic of the length l by the gravity force which is the analogue of Galileo experiment with dropping objects from the leaning tower of Pisa. Galileo have used two bodies made of the same material, differing only in size. The effects of air friction were ignored. The two bodies reached the ground at the same time. So, he supported the conclusion that the every falling body is falling with a uniform acceleration, the resistance of the medium being negligible. Galileo experimentation represented the kernel of scientific investigation and Galileo was keen to point this out (Frova et al., 2006).

Galileo experiment inspired Einstein in formulation of the equivalence principle with two reference frames, K and K'. K is a uniform gravitational field, whereas K' has no gravitational field but is uniformly accelerated in such a way that objects in the two frames experience identical forces. According to Einstein systems K and K' are physically exactly equivalent. (Einstein, 1911).

Or, in other words: *Inertia and gravity are identical; hence and from the results of special relativity theory it inevitably follows that the symmetric fundamental tensor* $g_{\mu\nu}$ *determines the metric properties of space, of the motion of bodies due to inertia in it, and, also, the influence of gravity* (Einstein, 1918).

According to Fock (1964), *principle of equivalence is understood to be the statement that in some sense a field of acceleration is equivalent to a gravitational field. It means that by introducing a suitable system of coordinates (which is usually interpreted as an accelerated frame of reference) one can so transform the equations of motion of a mass point in a gravitational field that in this new system they will have the appearance of equations of motion of a free mass point. Thus a gravitational field can, so to speak, be replaced, or rather imitated, by a field of acceleration. Owing to the equality of inertial and gravitational mass such a transformation is the same for any value of the mass of the particle. But it will succeed in its purpose only in an infinitesimal region of space, i.e. it will be strictly local. In the general case the transformation described corresponds mathematically to passing to a locally geodesic system of coordinates.*

The principle of equivalence states that it is impossible to distinguish between the action on a particle of matter of a constant acceleration, or, of static support in a gravitational field (Lyle, 2008).

We have seen that the motion of the accelerated string by the non-gravity forces differs from the motion of the string caused by the gravity with the acceleration g.

The controversions between different opinions can be easily solved with regard to the physical definition of gravity and inertia. Namely: gravity is form of matter in the physical vacuum. And inertia is the result of the interaction of the massive body with quantum vacuum being the physical medium.

It is well known that synchrotron radiation influences the motion of the electron in accelerators. The corresponding equation which describes the classical motion is so called the Lorentz-Dirac equation, which differs from the the so called Lorentz equation

$$mc\frac{du_\mu}{ds} = \frac{e}{c}F_{\mu\nu}u^\nu \tag{37}$$

only by the additional term which describes the radiative corrections. So, the equation with the radiative term is as follows (Landau et al., 1988):

$$mc\frac{du_\mu}{ds} = \frac{e}{c}F_{\mu\nu}u^\nu + g_\mu, \tag{38}$$

where u_μ is the four-velocity and the radiative term was derived by Landau et al. in the form (Landau et al., 1988):

$$g_\mu = \frac{2e^3}{3mc^3}\frac{\partial F_{\mu\nu}}{\partial x^\alpha}u^\nu u^\alpha - \frac{2e^4}{3m^2c^5}F_{\mu\alpha}F^{\beta\alpha}u_\beta + \frac{2e^4}{3m^2c^5}\left(F_{\alpha\beta}u^\beta\right)\left(F^{\alpha\gamma}u_\gamma\right)u_\mu. \tag{39}$$

The last equation can be easily converted into equation for charged particle moving in gravity. However, the term describing the radiation caused by gravity is not present

(Landau, et al., 1988).

It was proved by author (Pardy, 2009) that synchrotron radiation influences the spin motion of the electron in accelerators. The corresponding equation which describes the classical spin motion is so called the Bargmann-Michel-Telegdi-Pardy and is of the form (Pardy, 2009):

$$\frac{da_\mu}{ds} = 2\mu F_{\mu\nu}a^\nu - 2\mu' u_\mu F^{\nu\lambda}u_\nu a_\lambda +$$

$$\Lambda u_\mu \left\{ \frac{2e^3}{3mc^3}\frac{\partial F_{\lambda\nu}}{\partial x^\alpha}u^\nu u^\alpha - \right.$$

$$\left. \frac{2e^4}{3m^2c^5}F_{\lambda\alpha}F^{\beta\alpha}u_\beta + \frac{2e^4}{3m^2c^5}\left(F_{\alpha\beta}u^\beta\right)\left(F^{\alpha\gamma}u_\gamma\right)u_\lambda \right\} a^\lambda \tag{40}$$

where Λ is the bremsstrahlung constant.

Let us remark that the conversion of this equation to the situation where the interaction with the gravitational field is present, was not still derived.

We know, that free the fall law of the positronium is of the same law as the free fall of an electron, or, positron apart. Also, free fall of the protonium is of the same law as the free fall of the proton, or, antiproton apart. It was experimentally verified. It means that the charge interaction with gravity is zero. Gravity interact only with mass and the result of such interaction is the free fall with emission of gravitons. In case of the binary system it was confirmed by NASA and the spectral formula of the emission of gravitons by the binary was calculated by author (Pardy, 1983; 1994a; 1994b; 2011; 2018; 2019). In case of the existence of the gravitational index of refraction, the gravitational Cherenkov radiation is possible (Pardy, 1994c; 1994d).

While Galileo dropped objects from the leaning tower of Pisa, now, we have possibility to drop charged objects from the very high tower Burj Khalifa, in order to confirm the law that charged objects accelerated by the gravitational field do not radiate the electromagnetic energy. It is not excluded that such experiment with the adequate title Galileo-Pardy-Burj Khalifa project will be realized sooner, or, later. The project is cheaper than LHC.

4 The Zitterbewegung of Planets and Moon in the string model of gravity

4.1 Introduction

The string model of gravitational force was proposed by author 40 years ago (Pardy, 1980; 1996). In this model the string forms the mediation of the gravitational interaction between two gravitating bodies. It reproduces the Newtonian results in the first-order approximation and it predicts in the higher-oder approximations the existence of oscillations of the massive bodies interacting by the string. In case of the Moon it can be easily verified by NASA laser measurements.

It is well known from the history of physics that the problem of action-at-a-distance was for the first time seriously considered by Newton, in his letter to Bentley (Bentley, 1692), which is cited in "Principia Mathematica" (Newton, 1966) and discussed in the Stanford Encyclopedia of Philosophy (2006).

Instead of resolution of this problem Newton suggested the phenomenological theory of the gravitational force, where there exists no answer concerning the dynamics, or, the mechanism of action-at-a-distance. Newton himself was award that it necessary exists some mediation of interaction between two bodies at the different points in space because he has written (Newton. 1966): "*It is inconceivable, that inanimate brute matter, should without the mediation of something else which is not material, operate upon and affect other matter without mutual contact ..* ". In other words, the crucial notion in the Newton speculation is the mediation between two bodies. Now, we can say that Newton considered the string model of gravity because string model of gravity is the natural result of Newton scientific intuition.

The problem of mediation of the gravitational force between two bodies 1 and 2 can be analyzed by the way which forms the serious motivations for reconsidering the problem of action-at-a-distance. If we transmit body 1 with mass m from one point to other during very short time interval, then the gravitational force acting to the second body 2 with mass M necessary changes. However, in case that the second body is far from the first one, then much time elapses before it receives the gravitational input. The question is, where is the gravitational perturbation when the first body after short motion is yet in rest and the second one has no information on the motion on the first body?. It is evident the gravitational input is between body 1 and 2 on the line or string connecting body 1 with the body 2. The string is the Newton medium which transmits the gravitational force from one body to the other one.

It is well known that Newtonian theory is successful in its domain of validity, and general relativity is successful in accounting for the discrepancies between observed gravitational data and the Newtonian theory, as well as in resolving its problem of compatibility with special relativity. Nevertheless, Newton theory does not involve the string tension which is logical necessary as we have seen.

By analogy with the mechanical situation we will suppose the model where the attractive force between two bodies is transmitted as tension in the fictitious string connecting the one body with the another one. Then, the theoretical problem is to show that such model works and gives not only the old results but new results which cannot be derived from the original Newton law.

We will consider the string, the left end of which is fixed at the beginning of the coordinate system and mass m is fixed on the right end of the string. The motion of the system string and the body with mass m is the fundamental problem of the equations of the mathematical physics in case that the tension is linearly dependent on elongation (Tikhonov et al., 1977). We will show that it is possible to represent the Newton gravitational law by the string with the nonlinear tension in the string. Because of the strong nonlinearity of the problem the motion of the string and the body can be solved only approximately. In the following text, we will give the approximative solution of the classical two-body problem and then we obtain the string solution of this problem.

4.2 The classical two-body problem

Let us consider two bodies 1 and 2 with masses M and m, where $M \gg m$. The body 1 is supposed to be fixed at the origin of the coordinate system and the body 2 is for the simplicity moving in the interval

$$(R - \delta, R + \delta), \tag{1}$$

where $\delta \ll R$, which corresponds to the motion of planets of our Sun system. The Newton law

$$F = -\kappa \frac{Mm}{r^2}, \tag{2}$$

can be obviously expressed in the interval (1) approximately as

$$F \approx a\eta + b; \quad (-\delta, \delta) \ni \eta, \tag{3}$$

where

$$a = \frac{2\kappa Mm}{R^3}, \quad b = -\frac{\kappa Mm}{R^2}. \tag{4}$$

The motion of body 2 in the gravitational potential of body 1 is described by equation (Landau et al., 1965)

$$m\,\ddot{r} = -\kappa \frac{Mm}{r^2} + \frac{J^2}{mr^3}, \tag{5}$$

where J is the angular momentum of body 2. In the interval $(-\delta, \delta)$ we can write

$$r(t) = R + \eta(t) \tag{6}$$

and using approximation

$$\frac{1}{(R+\eta)^2} \approx \frac{1}{R^2}(1 - \frac{2\eta}{R}), \quad \frac{1}{(R+\eta)^3} \approx \frac{1}{R^3}(1 - \frac{3\eta}{R}), \tag{7}$$

we get after insertion of eq. (6) into eq. (5):

$$\ddot{\eta} + \omega^2\eta = \lambda, \tag{8}$$

where

$$\omega^2 = \frac{3J^2}{m^2R^4} - \frac{2\kappa M}{R^3} \tag{9}$$

$$\lambda = \frac{J^2}{m^2R^3} - \frac{\kappa M}{R^2}. \tag{10}$$

For the circle motion we have $J = m\omega R^2, r = R$ and from eqs. (9) and (10) it follows:

$$\omega = R^{-3/2}(\kappa M)^{1/2}; \quad \lambda = 0. \tag{11}$$

It is easy to see that the solution of eq. (8) is of the form:

$$\eta(t) = \Lambda\cos(\omega t + \vartheta) + \frac{\lambda}{\omega^2}, \tag{12}$$

where Λ and ϑ are constants involving the initial conditions of motion of the body 2.

So far we have supposed no dynamics of mediation of the interaction between body 1 and 2. However, only the model involving the mechanism of mediation of interaction can describe logically consistent reality and explain the Newton puzzle. Let us try to elaborate the consistent and realistic model which describes the mechanism of mediation.

4.3 The string mediation of interaction

In this section we will solve the motion of a body 2 at the end of the string on the assumption that the tension in the string is nonlinear and it generates the Newton law in the statical regime. We will give the rigorous mathematical formulation of such problem named the Newton-Pardy string mediation of interaction. While for the Hook tension the problem has solution by the Fourier method, in case of the nonlinear tension it is not possible to use this method.

There is no evidence about solution of this Newton-Pardy string problem in the textbooks of mathematical physics, or, in the mathematical journals. So, it seems, we solve this problem for the first time.

Let be given the string, the left end of which is fixed at beginning and the right end is at point l at the state of equilibrium. The deflection of the string element dl at point x and time t let be $u(x,t)$ where $x \in (0,l)$ and

$$\eta(t) = u(l,t), \quad \eta(0) = u(l,0). \tag{13}$$

Then, the motion of body 2 is described by the motion of the right end-point of the string, when the left point is constantly fixed at the origin.

The differential equation of motion of string elements can be derived by the following way: We suppose that the force acting on the element dl of the string is given by the law:

$$T(x,t) = ES\left(\frac{\partial u}{\partial x}\right)^{-2}, \tag{14}$$

where E is the modulus of elasticity, S is the cross section of the string. We easily derive that

$$T(x+dx) - T(x) = 2ES\,(u_x)^{-3}\,u_{xx}dx. \tag{15}$$

The mass dm of the element dl is $\varrho ESdx$, where ϱ is the mass density of the string matter and the dynamical equilibrium gives

$$\varrho Sdxu_{tt} = 2ESu_{xx}\,(u_x)^{-3}\,dx. \tag{16}$$

Putting

$$\varrho = \varrho_0\frac{2}{(u_x)^3}; \quad \varrho_o = const., \tag{17}$$

we get

$$\frac{1}{c^2}u_{tt} - u_{xx} = 0; \quad c = \left(\frac{E}{\varrho_0}\right)^{1/2}. \tag{18}$$

The last procedure was performed evidently in order to get the wave equation.

Now, let us look for the correspondence between the string tension and the Newton law. Putting $u_{tt} = 0$ we get the stationary case with the solution

$$u(x,t) = \alpha x + \beta. \tag{19}$$

Because $u(0,t) \equiv 0$, we get $\beta = 0$. Then $u_x(x,t) = \alpha$ is not dependent on x and according to the definition of the tension the force is constant along the length of the string which is the same result as in the case with the Hook law.

For sufficiently big elongation we have $u(l) \gg l$ and the elongation at point l is the distance of the right end of the string from the origin and it means that the force acting on the right end of the string is proportional to the minus square of the distance of the right end of the string as in the Newton gravitational law. So, we have demonstrated that the Newton gravitational force can be realized by the string, while the Newton original force involves no mediation between two bodies. Now, we can repeat the formulation of the problem described in the previous section in such a way that we will use the dynamical equation (18) instead of eq. (5). So, let us approach the solution of the problem of the motion of body on the end of the string where the tension of the string is defined by equation (14).

From (19) we have:

$$\alpha = \frac{u(l,t)}{l}. \tag{20}$$

Thus,

$$T(l,t) = -\frac{ESl^2}{u^2(l,t)} = -\kappa \frac{mM}{u^2(l,t)}, \tag{21}$$

which gives the relation between the string constants and the gravitating parameters

$$ESl^2 = \kappa m M. \tag{22}$$

The complete solution of eq. (18) includes the initial and boundary conditions. The simplest nontrivial initial conditions can be chosen with regard to the character of the problem and they are:

$$u(x,0) = \frac{R}{l}x, \quad u_t(x,0) = 0. \tag{23}$$

The boundary conditions are given with respect to the dynamical equation (5):

$$u(0,t) = 0, \quad m u_{tt}(l,t) = T(l,t) + \frac{J^2}{m u^3(l,t)}. \tag{24}$$

The solution of the wave equation with the strongly nonlinear boundary conditions is evidently beyond the possibility of the present mathematical physics. Nor the Fourier method, nor the d'Alembert one can be used in solution of our problem. So we are forced to find only the approximation of this problem. For this goal we write:

$$u(x,t) = \frac{R}{l}x + v(x,t), \tag{25}$$

from which follows

$$u_x(x,t) = \frac{R}{l} + v_x, \quad u(l,t) = R + v \tag{26}$$

and we suppose that $v \ll R$. In such a way the initial conditions are:

$$v(x,0) = 0, \quad v_t(x,0) = 0. \tag{27}$$

The approximative formulas are given in the following form:

$$\frac{1}{u_x^2(x,t)} \approx \frac{l^2}{R^2} - \frac{2v_x l^3}{R^3}, \tag{28}$$

$$\frac{1}{u^3(x,t)} \approx \frac{1}{R^3} - \frac{3v}{R^4}. \tag{29}$$

So, we get the new problem of mathematical physics: the wave equation

$$v_{tt} = c^2 v_{xx} \tag{30}$$

with the initial conditions

$$v(x,0)=0; \quad v_t(x,0)=0 \tag{31}$$

and with the boundary conditions

$$v(0,t)=0; \quad mv_{tt}(l,t)=a+bv_x(l,t)+dv(l,t), \tag{32}$$

where we have put

$$a=-\kappa\frac{Mm}{R^2}+\frac{J^2}{mR^3}; \quad b=\frac{2\kappa Mm}{R^3}l; \quad d=-\frac{3J^2}{mR^4}. \tag{33}$$

The equation (30) with the initial and boundary conditions (31) and (32) represents one of the standard problems of the mathematical physics and can be easily solved using the Laplace transform (Arfken, 1967):

$$\hat{L}u(x,t)\stackrel{d}{=}\int_0^\infty e^{-pt}u(x,t)dt\stackrel{d}{=}u(x,p). \tag{34}$$

Using (30) we get with $\hat{L}v(x,t)\stackrel{d}{=}\varphi(x,p)$:

$$\hat{L}v_{tt}(x,t)=p^2\varphi(x,p)-pv(x,0)-v_t(x,0)=p^2\varphi(x,p) \tag{35}$$

$$\hat{L}v_{xx}(x,t)=\varphi_{xx}(x,p); \quad \hat{L}a=\frac{a}{p}; \quad \hat{L}v(0,t)=\varphi(0,p)=0. \tag{36}$$

After elementary mathematical operations we get the differential equation for φ in the form:

$$\varphi_{xx}(x,p)-k^2\varphi(x,p)=0; \quad k=p/c. \tag{37}$$

with the boundary condition in eq. (36).

We are looking for the the solution of eq. (37) in the form

$$\varphi(x,p)=c_1\cosh kx+c_2\sinh kx. \tag{38}$$

We get from the boundary conditions in eq. (36) $c_1=0$ and

$$c_2=\frac{a}{p}\,\frac{1}{(mp^2-d)\sinh kl-bk\cosh kl}. \tag{39}$$

The corresponding $\varphi(x,p)$ is of the form:

$$\varphi(x,p)=\frac{a}{p}\,\frac{\sinh kx}{(mp^2-d)\sinh kl-bk\cosh kl} \tag{40}$$

The corresponding function $v(x,t)$ follows from the theory of the Laplace transform as the mathematical formula:

$$v(x,t) = \frac{1}{2\pi i}\oint e^{pt}\varphi(x,p)dp = \sum_{p=p_n} res\ e^{pt}\varphi(x,p) =$$

$$\sum_{p=p_n} res\ e^{pt}\frac{a}{p}\ \frac{\sinh kx}{(mp^2-d)\sinh kl - bk\cosh kl}, \tag{41}$$

where p_n are poles of the function $\varphi(x,p)$ and they are evidently given by equation

$$\Big[(mp^2-d)\sinh kl - bk\cosh kl\Big] = 0, \tag{42}$$

which is equivalent with $k \to ik$ to

$$\tan kl = \frac{-bk}{mc^2k^2+d}. \tag{43}$$

In case of $k \ll 1$ we have two solutions: $p_0 = 0$ and

$$p_{1/2} = \pm\left(\frac{3J^2}{m^2R^4} - \frac{2\kappa M}{R^3}\right)^{1/2}, \tag{44}$$

which is in agreement with eq. (9) obtained by the approximation of classical Kepler problem. Further we have got the oscillations with frequencies p_n in the higher order approximation:

$$p_n \quad \to \quad \frac{n\pi}{l}, \quad n \gg 1 \tag{45}$$

At present time it is still question, how to detect these oscillations, or, if it is possible to use the experimental procedures of Braginskii et al. (1977) for the detection. The analogous situation was in quantum physics, where the zero frequencies of vacuum was considered as meaningless till it was shown by Casimir that they give the attractive force between two conductive plates. It is not excluded that the Zitterbewegung of celestial bodies will be confirmed by NASA laser experiments.

4.4 Discussion

The basic heuristic idea of this article was the string realization of the gravitational force between two bodies.

In order to realize this idea we introduced the string of the length l with the nonlinear tension which generates in the statical situation the Newton law at the distances much greater then is the fundamental length of the string. We have solved this problem only approximately because at present time the exact solution is beyond possibilities of mathematics.

While the string with the Hook tension has the equilibrium state, our string is in the dynamical state.

The difficulties with the action-at-distance as is an indication of the limitation of the Newtonian theory. The general relativity solves the problem as the metric theory of the gravitational interaction. General relativity is the geometry theory of space time and as such it is the physical model based on the Riemann and Gauss ideas. In General relativity, there is the fundamental notion, the metric tensor, while in the Newton theory the basic building stone is force. In our theory, which is the dynamical version of the Hook theory of string is the basic notion the tension of the string.

The string is between any two masses and it means that universe is occupied by strings, vacuum and by bodies and particles. Our planet is for instance connected by strings with all stars in universe.

Our problem was never defined to our knowledge in the mathematical or physical textbooks, monographies or scientific journals. Thus, our approach is original.

The proposed model can be also related in the modified form to the problem of the radial motion of quarks bound by a string and used to calculate the excited states of such system. The original solution was considered by Bardeen et al. (1976; 1976) Chodos et al. (1974) and by Frampton (1975). The new analysis of such problem was performed by Nesterenko (1990) and author (Pardy, 2016). So, there are open way in particle physics to follow our approach. It is not excluded that the Zitterbewegung of moon will be confirmed by NASA laser systems (NASA first!).

5 The pulse propagation in the string with the massive ends

5.1 Introduction

We consider the elastic rod of a large mass M, the left end of which is fixed to a body of mass $m << M$ and the second body of mass m is fixed to the right end of the rod. The force of the delta-function form is applied to the left side of the rod. We find the propagation of the pulse in the system. Our problem represents the missing problem in the textbooks on mechanics. The relation of our theory to the quark-string model of mesons is evident.

The determination of the hadron mass spectrum in the framework of quantum chromodynamics (QCD), or dynamical states of mesons still remains a unsolved problem. So, the potential methods and the string methods are used for this purpose.

In the string model of hadrons the quarks are treated to be tied together by a gluon tube which can be approximated by the tube of vanishing width, or by string (Nambu, 1974). Then, dynamics of hadrons can be approximated by the Nambu-Goto action for the relativistic string. The merit of the string model is the natural explanation of the quark confinement and the dynamics of the system. On the other hand there are some mathematical problems when the quark masses are different from zero. So, there are many trials to give the final words to the problem of hadron dynamics and hadron masses. It is well known that some models of string theory involves also the so called extra-dimension.

However, it is well known that this theory is at present time not predictable. The goal of theoretical physics is predictions based on the mathematical knowledge. On the other hand the goal of the experimental physic is the confirmation of theory with appropriate experimental simplicity or virtuosity.

We know that the starting point of string approach to the two-body problem with massive quarks at the ends of string is based on the following action (Nesterenko, 1990; Barbashov et al., 1990):

$$S = -\gamma \int_{t_1}^{t_2} dt \int_{\sigma_1(t)}^{\sigma_2(t)} d\sigma \sqrt{\mathbf{x}'^2(1 - \dot{\mathbf{x}}^2) + (\dot{\mathbf{x}}\mathbf{x}')^2} \quad -$$

$$\sum_{a=1}^{2} m_a \int_{t_1}^{t_2} dt \sqrt{1 - \left(\frac{d\mathbf{x}_a}{dt}\right)^2}, \qquad (1)$$

where the space-like string coordinates $\mathbf{x}$ are parametrized by variable σ numbering the points along the string (the so called $t = \tau$ gauge) $\mathbf{x} = \mathbf{x}(t, \sigma)$ (Barbashov and Nesterenko, 1990). Here $\mathbf{x}_a = \mathbf{x}(t, \sigma_a), a = 1, 2; \dot{\mathbf{x}} = \partial\mathbf{x}/\partial t, \mathbf{x}' = \partial\mathbf{x}/\partial\sigma, \gamma$ is the string tension with dimension (length)$^{-2}$.

From this action the equation of motion for the rotating string can be derived and it is possible to show that this model is integrable. However there some problems with this model as was shown by Nesterenko (1990).

We use here different approach to the string dynamics which consists in experimental situation that we consider the pulse applied to the left side of a string and then determine the string dynamics. We do not consider in this simplification the rotating string.

So, let us consider the elastic rod (or, string) of large mass M, the left end of which is joined with mass $m << M$ and body of mass m is fixed to the right end of the rod (Pardy, 2005). The force of the delta-function form is applied to the left side of the rod. The delta-function is chosen for simplicity. This function can be replaced by the different functions. We show that the internal motion of the elastic rod medium is controlled by the wave equation. We show that the momentum of the massive ends is not conserved in time. Our problem represents the missing problem in textbooks on mechanics.

The pedagogical benefit of this article is in the rigorous definition of the problem in the university mechanics and in the proof that the Dirac delta-function elegantly works in mechanics.

The experimental demonstration of the delta-function tension can be easily performed as it is described in the text. The mathematical formulation of the problem and the solution gives deep insight in the dynamics of strings and rods with the massive ends. We give new possibilities of the university mechanics. We hope that our approach can form the serious motivation for study the string theory on LHC.

5.2 Classical particle interaction with an impulsive force

Let us first show that use of the impulsive force of the delta-function form is physically meaningful in a classical mechanics of a point particle. We idealize the impulsive force by the Dirac δ-function.

Newton's second law in the one-dimensional form for the interaction of a massive particle with mass m with force F is $ma = F$. Then, with F being an impulsive force $P\delta(\alpha t)$ it is as follows:

$$m\frac{d^2x}{dt^2} = P\delta(\alpha t), \tag{2}$$

where P and α are some constants, with MKSA dimensionality $[P] = \text{kg.m.s}^{-2}$, $[\alpha] = \text{s}^{-1}$. We put $|\alpha| = 1$.

Using the Laplace transform (Arfken, 1967) in the last equation, with

$$\int_0^\infty e^{-st}x(t)dt \stackrel{d}{=} X(s), \tag{3}$$

$$\int_0^\infty e^{-st}\ddot{x}(t)dt = s^2X(s) - sx(0) - \dot{x}(0), \tag{4}$$

$$\int_0^\infty e^{-st}\delta(\alpha t)dt = \frac{1}{\alpha}, \tag{5}$$

we obtain:

$$ms^2X(s) - msx(0) - m\dot{x}(0) = P/\alpha. \tag{6}$$

For a particle starting from the rest with $\dot{x}(0) = 0, x(0) = 0$, we get

$$X(s) = \frac{P}{ms^2\alpha}. \tag{7}$$

Using the inverse Laplace transform, we obtain

$$x(t) = \frac{P}{m\alpha}t \tag{8}$$

and

$$\dot{x}(t) = \frac{P}{m\alpha}. \tag{9}$$

Let us remark, that the Laplace transform of the delta-function in eq. (5) is absolutely rigorous if we use first the Laplace transform of $\delta(\alpha(t-\varepsilon))$ and then we take $\varepsilon = 0$.

In case of the harmonic oscillator with the damping force and under influence of the general force $F(t)$, the Newton law is as follows:

$$m\frac{d^2x(t)}{dt^2} + b\dot{x}(t) + kx(t) = F(t). \tag{10}$$

After application of the Laplace transform (3) and with regard to the same initial conditions as in the preceding situation, $\dot{x}(0) = 0, x(0) = 0$, we get the following algebraic equation:

$$ms^2X(s) + bsX(s) + kX(s) = F(s), \tag{11}$$

or,

$$X(s) = \frac{F(s)}{m\omega_1}\frac{\omega_1}{(s + b/2m)^2 + \omega_1^2} \tag{12}$$

with $\omega_1^2 = k/m - b^2/4m^2$.

Using inverse Laplace transform denoted by symbol $\mathcal{L}^{-1}$ applied to multiplication of functions $f_1(s)f_2(s)$,

$$\mathcal{L}^{-1}(f_1(s)f_2(s)) = \int_0^t d\tau F_1(t-\tau)F_2(\tau), \tag{13}$$

we obtain with $f_1(s) = F(s)/m\omega_1$, $f_2(s) = \omega_1/((s + b/2m)^2 + \omega_1^2)$, $F_1(t) = F(t)/m\omega_1, F_2(t) = \exp(-bt/2m)\sin\omega_1 t$.

$$x(t) = \frac{1}{m\omega_1}\int_0^t F(t-\tau)e^{-\frac{b}{2m}\tau}\sin(\omega_1\tau)d\tau. \tag{14}$$

For impulsive force $F(t) = P\delta(\alpha t)$, we have from the last formula

$$x(t) = \frac{(P/\alpha)}{m\omega_1}e^{-\frac{b}{2m}t}\sin\omega_1 t. \tag{15}$$

5.3 The pulse propagating in a rod

In this section we will solve the motion of a string or rod with the massive ends (the body with mass m is fixed to the every end of the string) on the assumption that the tension in the string is linear and the applied force is of the Dirac delta-function. First, we will derive the Euler wave equation from the Hook law of tension and then we will give the rigorous mathematical formulation of the problem. Linearity of the wave equation enables to solve this problem by the Laplace transform method. We follow monograph by Tikhonov et al. (1977) and the author preprint (Pardy, 1996) where this method was used to solve the Gassendi model of gravity. Although Gassendi (Fraser et al., 1998) is known in physics as the founder of the modern atomic theory of matter, his string model of gravity was not accepted. The Newton reaction to this model was empirical. He said: "Hypotheses non fingo". It seems that Gassendi ideas was applied later by Faraday in his theory of electromagnetism.

Let us remark, that Gassendi solved the philosophical question: where is the force between body A and B, when the intermediate distance is L? The answer is, that it is hidden in the straight-line string connecting A and B. The tension is given by the Newton law and it is constant along the string. This is a consequence of the nonlinear Hook law. In case of the planetary situation, the motion of a planet around Sun is performed along an ellipse which is slightly undulated, or, wavy. The undulation is so small, that it cannot be observed by any the most modern laser technique.

The present problem can be also defined as a central collision of two bodies (balls). While in the basic mechanics the central collision is considered as a contact collision of the two balls, here, the collision is mediated by the string or rod.

To our knowledge, the present problem is not involved in the textbooks of mathematical physics or in the mathematical journals. This problem was not possible to define and solve in the Newton period, because the method of solution is based on the Euler partial wave equation, the Laplace transform, The Riemann-Mellin transform, the Bromwich integral and Bromwich contour and other ingredients of the operator calculus which was elaborated after the Newton period.

Now, let us consider the rod (or string) of the length L, the left end of which is joined with mass m and the right end is joined with mass m. The force of the delta-function form is applied to the left end and the initial state of the rod is the sate of equilibrium. The deflection of the rod element dx at point x and time t let be $u(x,t)$ where $x \in (0, L)$.

The differential equation of motion of string elements can be derived by the following way (Tikhonov et al., 1977). We suppose that the force acting on the element dx of the string is given by the law:

$$T(x,t) = ES\left(\frac{\partial u}{\partial x}\right), \tag{16}$$

where E is the modulus of elasticity, S is the cross section of the string. We easily derive that

$$T(x+dx) - T(x) = ESu_{xx}dx. \tag{17}$$

The mass dm of the element dx is $\varrho ESdx$, where $\varrho = const$ is the mass density of the string matter and the dynamical equilibrium gives

$$\varrho Sdxu_{tt} = ESu_{xx}dx. \tag{18}$$

So, we get

$$\frac{1}{c^2}u_{tt} - u_{xx} = 0; \quad c = \left(\frac{E}{\varrho}\right)^{1/2}. \tag{19}$$

Now, we get the problem of the mathematical physics in the form of the differential wave equation:

$$u_{tt} = c^2 u_{xx} \tag{20}$$

with the initial conditions

$$u(x,0) = 0; \quad u_t(x,0) = 0 \tag{21}$$

and with the boundary conditions

$$mu_{tt}(0,t) = au_x(0,t) + P\delta(\alpha t); \quad mu_{tt}(L,t) = au_x(L,t), \tag{22}$$

where we have put

$$a = -ES; \quad P = \text{some constant}. \tag{23}$$

The delta-function can be approximatively realized by the strike of the hammer to the left end of the rod.

The equation (20) with the initial and boundary conditions (21) and (22) represents one of the standard problems of the mathematical physics and can be easily solved using the Laplace transform (Arfken, 1967):

$$\hat{L}u(x,t) \stackrel{d}{=} \int_0^\infty e^{-pt}u(x,t)dt \stackrel{d}{=} \varphi(x,p). \tag{24}$$

Using (24) and (20) we get:

$$\hat{L}u_{tt}(x,t) = p^2\varphi(x,p) - pu(x,0) - u_t(x,0) = p^2\varphi(x,p), \tag{25}$$

$$\hat{L}u_{xx}(x,t) = \varphi_{xx}(x,p); \quad \hat{L}\delta = 1/\alpha; \quad \hat{L}u(0,t) = \varphi(0,p) = 0. \tag{26}$$

After elementary mathematical operations we get the differential equation for φ in the form

$$\varphi_{xx}(x,p) - k^2\varphi(x,p) = 0; \quad k = p/c. \tag{27}$$

with the boundary condition in eq. (22).

We are looking for the the solution of eq. (27) in the form

$$\varphi(x,p) = c_1 \cosh kx + c_2 \sinh kx. \tag{28}$$

We get from the boundary conditions in eq. (22)

$$c_1 = \frac{1}{p}\,\frac{ac(P/\alpha)\cosh(pL/c) - (P/\alpha)mpc^2\sinh(pL/c)}{\sinh(pL/c)(a^2 - m^2p^2c^2)}, \tag{29}$$

$$c_2 = -\frac{(P/\alpha)c}{ap} + \frac{(P/\alpha)mac^2\cosh(pL/c) - (P/\alpha)pm^2c^3\sinh(pL/c)}{a\sinh(pL/c)(a^2 - m^2c^2p^2)}. \tag{30}$$

The corresponding $\varphi(x,p)$ is of the form:

$$\varphi(x,p) = \frac{1}{p}\,\frac{ac(P/\alpha)\cosh(pL/c) - (P/\alpha)mpc^2\sinh(pL/c)}{\sinh(pL/c)(a^2 - m^2p^2c^2)}\cosh(px/c) \quad +$$

$$\left[-\frac{(P/\alpha)c}{ap} + \frac{a(P/\alpha)mc^2\cosh(pL/c) - bpm^2c^3\sinh(pL/c)}{a\sinh(pL/c)(a^2 - m^2c^2p^2)}\right]\sinh(px/c). \tag{31}$$

The corresponding function $u(x,t)$ follows from the theory of the Laplace transform as the mathematical formula (res is residuum)(Arfken, 1967):

$$u(x,t) = \frac{1}{2\pi i}\oint e^{pt}\varphi(x,p)dp = \sum_{p=p_n} \text{res } e^{pt}\varphi(x,p) =$$

$$\sum_{p=p_n} \text{res } e^{pt}\frac{1}{p}\,\frac{ac(P/\alpha)\cosh(pL/c)}{\sinh(pL/c)(a^2 - m^2p^2c^2)}\cosh(px/c) \quad -$$

$$\sum_{p=p_n} \text{res } e^{pt}\frac{(P/\alpha)mc^2}{(a^2 - m^2p^2c^2)}\cosh(px/c) \quad -$$

$$\sum_{p=p_n} \text{res } e^{pt}\left[\frac{(P/\alpha)c}{ap}\right]\sinh(px/c) \quad +$$

$$\sum_{p=p_n} \text{res } e^{pt}\left[\frac{m(P/\alpha)c^2\cosh(pL/c)}{\sinh(pL/c)(a^2 - m^2p^2c^2)}\right]\sinh(px/c) \quad -$$

$$\sum_{p=p_n} \text{res } e^{pt}\left[\frac{(P/\alpha)pm^2c^3}{a}\,\frac{1}{(a^2 - m^2c^2p^2)}\right]\sinh(px/c) \quad =$$

$$u_1 - u_2 - u_3 + u_4 - u_5, \tag{32}$$

where

$$u_j = \sum \text{res } e^{pt}\frac{A_j}{B_j}; \quad j = 1,2,3,4,5 \tag{33}$$

and

$$A_1 = ac(P/\alpha)\cosh(pL/c)\cosh(px/c); \quad B_1 = p\sinh(pL/c)(a^2 - m^2p^2c^2) \tag{34}$$

$$A_2 = (P/\alpha)mc^2\cosh(px/c); \quad B_2 = (a^2 - m^2p^2c^2) \tag{35}$$

$$A_3 = (P/\alpha)c\sinh(px/c); \quad B_3 = ap \tag{36}$$

$$A_4 = (P/\alpha)mc^2\cosh(pL/c)\sinh(px/c); \quad B_4 = \sinh(pL/c)(a^2 - m^2p^2c^2) \tag{37}$$

$$A_5 = (P/\alpha)pm^2c^3\sinh(px/c); \quad B_5 = a(a^2 - m^2p^2c^2). \tag{38}$$

We know from the theory of the complex functions that if the pole of some function $f(z)/g(z)$ is simple and it is at point a, then the residuum is as follows (Arfken, 1967):

$$\text{residuum} = \frac{f(a)}{g'(a)}. \tag{39}$$

If the pole at point a of the function $f(z)$ is multiple of the order m, then the residuum is defined as follows:

$$\text{residuum} = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z) \right]. \tag{40}$$

Let us first determine the function

$$u_1 = \sum \text{res } e^{pt} \frac{A_1}{B_1}. \tag{41}$$

Poles of B_1 are at points $p = 0$, (this is a pole of the order 2), $p = +a/mc, p = -a/mc$ and $p_n = +i\pi nc/L, p_n = -i\pi nc/L, n = 1, 2, 3, \ldots$. So, the function u_1 is as follows:

$$u_1 = \frac{(P/\alpha)c^2}{La} t - \frac{(P/\alpha)c}{a} \cosh\left(\frac{aL}{mc^2}\right) \cosh\left(\frac{ax}{mc^2}\right) \sinh\left(\frac{at}{mc}\right) \quad +$$

$$\sum_{n=1}^{n=\infty} \frac{2a(P/\alpha)c}{\pi n} \frac{L^2}{a^2L^2 + m^2\pi^2 n^2 c^4} \cos\left(\frac{\pi n x}{L}\right) \sin\left(\frac{\pi n c t}{L}\right). \tag{42}$$

.

For the function u_2 we get:

$$u_2 = \left(-\frac{(P/\alpha)c}{a}\right) \sinh\left(\frac{at}{mc}\right) \cosh\left(\frac{ax}{mc^2}\right). \tag{43}$$

$$u_3 = 0. \tag{44}$$

For u_4 and u_5 we get:

$$u_4 = \left(-\frac{(P/\alpha)c}{a}\right) \coth\left(\frac{aL}{mc^2}\right) \sinh\left(\frac{ax}{mc^2}\right) \sinh\left(\frac{at}{mc}\right) \tag{45}$$

$$u_5 = \left(-\frac{(P/\alpha)c}{a}\right) \sinh\left(\frac{ax}{mc^2}\right) \sinh\left(\frac{at}{mc}\right) \tag{46}$$

The dimensionality of u is $[u] = \text{m}$ and $u(x, 0) = 0$. The momentum of a left particle $p = m\dot{u}(0, t)$, or right particle $p = m\dot{u}(L, t)$ is not conserved. Only the total momentum of a system is conserved.

5.4 Experiment

The propagation of a pulse in one direction is possible to confirm experimentally by using the heavy elastic rod (the segment of a rail). The delta-form force (tension) can be generated approximately by the strike of hammer. This pulse causes the motion of some ball touching the opposite side to the hammer strike on the rod.

The electronic verification of the propagation of a pulse in the string and by the computer elaboration is not problem of the electronic and computer laboratory. It can be performed using so called tensometer applying to the rod. At the same time it is possible to realize the optical verification of our model using the infrared rays reflected by the element of a string or rod, because the phonons forming the pulse can be absorbed by the infrared rays. So, we see that our mathematical model of a string can be considered as an integral part of electronics and of quantum solid state physics.

Now, let us give some general ideas following from the wave equation. It is well known that the solution of this equation is in general in the form (Landau et al., 2000):

$$f\left(t - \frac{x}{c}\right); \quad g\left(t + \frac{x}{c}\right), \tag{47}$$

where functions f, g are general. It means it involves also the function of the delta-form. For the wave propagating from the left side to the right side, we take function f. The corresponding tension in the rod is

$$T = ESu_x(x,t) = ESf'\left(t - \frac{x}{c}\right)\left(\frac{-1}{c}\right). \tag{48}$$

We easily see that $T(x = 0, t = 0) = T(x = L, t = L/c)$, and it means that when the pulse force is created at the left end of the rod then it propagates in the rod and after time L/c it is localized in the right end of the rod.

5.5 Discussion

Our article is the modification of the recent author article (Pardy, 2005), which is reformulation and elaboration of some problems involved in the textbooks on mathematical physics. However, our approach is pedagogically original in the sense that we use the initial force of a delta-function form to show the internal motion of the string or rod. The delta-function form of electromagnetic pulse was used also by author (Pardy, 2002; 2003) to discuss the quantum motion of an electron in the laser pulse. We have considered here the real strings and rods in the real space and we do not use extra-dimensions and unrealistic strings. The M-dimensional geometrical object cannot be realized in N-dimensional space for $M > N$ (Pardy, 2004).

Our problem with the real strings and rods can be generalized for the two-dimensional and three-dimensional situation. It can be also generalized to the situation with the dissipation of waves in the strings and rods. In this case it is necessary to write the wave equation with the dissipative term and then to solve this problem *ab initio*.

The linear string model can be generalized to the nonlinear strings, strings with the internal structure and variable cross-section, the magnetic strings, the dielectric strings, or the strings can be considered as the linear chains composed from the massive elements. Then, we can define the Born-Kárman chain, Heisenberg chain, Bethe chain, Ising chain, Thirring chain, and many others quantum chains.

It seems, there is no information in textbooks on mechanics on the central collision of two particle where the force is mediated by string or rod. Similarly, there is not the solution of our problem in the famous monograph by Pars (1964). So, this is the missing problem in the textbooks on mechanics.

The proposed model with the string with massive ends can be also related in the modified form to the recent problems of the radial motion of quarks bound by a strings, so called quark-string model of mesons and used to calculate the excited states of such system. The vibration energy of such states are involved in formula (42). The quantum version is necessary to elaborate. The recent analysis of such problem was performed by Lambiase and Nesterenko (1996) and Nesterenko and Pirozhenko (1997), and others. We hope that our approach and their approach will be unified to generate the new way of the string theory of matter and space-time.

6 The quantum string motion under the periodic boundary conditions

6.1 Introduction

We consider the string, the left end of which is fixed and the right end of the string is in periodic motion. We derive the quantum internal motion of this system.

According to Nielsen and Olesen (1973) there is parallelism between the Higgs model of broken gauge invariance and the Landau-Ginzburg superconductivity theory on the one hand and and the dual string model and at the same time between Abrikosov flux lines in superconductors II. So dual string is mathematical realization of magnetic flux tube in equilibrium againts the pressure of the surrounding charged superfluid. Only strings with no ends were considered by them (Nambu, 1974). The internal quantum motion of strings is not considered by the authors.

We consider here the string, the left end of which is fixed and the right end of the string is in periodic motion. We derive the quantum internal motion of this system. We use the so called the oscillator quantization of string.

The non-relativistic quantization of the equation for the energy of a free particle

$$\frac{p^2}{2m} = E \tag{1}$$

consists in replacing classical quantities by operators. We get the non-relativistic Schrödinger equation for a free particle. The operator replacings are $E \rightarrow i\hbar\frac{\partial}{\partial t}$, $\mathbf{p} \rightarrow -i\hbar\nabla$.

The Schrödinger equation suffers from not being relativistically covariant, meaning it does not take into account Einstein's special relativity.

It is natural to perform the special relativity generalization of the energy relation describing the energy:

$$E = \sqrt{p^2c^2 + m^2c^4}. \tag{2}$$

Then, just inserting the quantum mechanical operators for momentum and energy yields the equation

$$i\hbar\frac{\partial}{\partial t} = \sqrt{(-i\hbar\nabla)^2c^2 + m^2c^4}. \tag{3}$$

This, however, is a cumbersome expression to work with because the differential operator cannot be evaluated while under the square root sign.

Klein and Gordon instead began with the square of the above identity, i.e. $E^2 = p^2c^2 + m^2c^4$, which, when quantized, gives

$$\left(i\hbar\frac{\partial}{\partial t}\right)^2 = (-i\hbar\nabla)^2c^2 + m^2c^4. \tag{4}$$

So, we have seen that the quantization of classical mechanics is the simple replacing classical quantities by operators. We use here the novel quantization method where classical oscillators forming the classical systems are replaced simply by the quantum solution of quantum oscillators. The natural step is to apply the method to motion of the classical string.

6.2 The classical derivation of the string motion

The differential equation of motion of string elements can be derived by the following way (Tikhonov et al., 1977). We suppose that the force acting on the element dx of the string is given by the law:

$$T(x,t) = ES\left(\frac{\partial u}{\partial x}\right), \tag{5}$$

where E is the modulus of elasticity, S is the cross section of the string. We easily derive that

$$T(x+dx) - T(x) = ESu_{xx}dx. \tag{6}$$

The mass dm of the element dx is $\varrho ESdx$, where $\varrho = const$ is the mass density of the string matter and the dynamical equilibrium gives

$$\varrho S dx u_{tt} = ES u_{xx} dx. \tag{7}$$

So, we get

$$\frac{1}{c^2} u_{tt} - u_{xx} = 0; \quad c = \left(\frac{E}{\varrho}\right)^{1/2}. \tag{8}$$

Now, let us consider the following problem of the mathematical physics. The right end of the string is in the periodic motion $u(l, 0) = A \sin(\omega t + \varphi)$. So we solve the mathematical problem:

$$u_{tt} = c^2 u_{xx} \tag{9}$$

with the initial conditions

$$u(x, 0) = 0; \quad u_t(x, 0) = 0 \tag{10}$$

and with the boundary conditions

$$u(0, t) = 0; \quad u(L, t) = A \sin(\omega t + \varphi). \tag{11}$$

The equation (10) with the initial and boundary conditions (10) and (11) represents one of the standard problems of the mathematical physics and can be easily solved using the the standard methods . The solution is elementary (Lebedev et al., 1955) and it is the integral part of equations of mathematical physics (Tikhonov et al., 1977):

$$u(x, t) = \frac{Ac}{ES} \left[\frac{\sin \frac{\omega x}{c}}{\frac{\cos \omega l}{c}}\right] \sin(\omega t + \varphi). \tag{12}$$

So, we see that the string motion is a such that at every point $X \in (0, l)$ there is an oscillator with an amplitude

$$\mathcal{A} = \frac{Ac}{ES} \left[\frac{\sin \frac{\omega X}{c}}{\frac{\cos \omega l}{c}}\right]; \quad X \in (0, l). \tag{13}$$

6.3 Quantization of the string motion by the harmonic oscillators

It is well known that harmonic oscillator equation

$$\ddot{x} + \omega^2 x = 0; \quad \omega = \sqrt{k/m} \tag{14}$$

has the solution

$$x(t) = A \cos(\omega t + \varphi). \tag{15}$$

In case of the quantum mechanical oscillator motion, the solution for the stationary sates is (Grashin, 1974)

$$\psi_n = N_n H_n \exp\left(-\xi^2/2\right); \quad \xi = x\sqrt{m\omega/\hbar}, \tag{16}$$

where N_n is the normalization constant

$$N_n = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{1}{2^n n!}} \tag{17}$$

and H_n are the Hermite polynomials defined by the following relation

$$H_n = (-1)^n e^{\xi^2} \frac{d^n}{d\xi^n} \exp\left(-\xi^2/2\right). \tag{18}$$

So, the wave function of the one oscillator of the string with the periodic end in the form $\psi_i(x_i - x, t)$):

$$\psi_i(x,t) = \frac{Ac}{ES} N_{n_i} \left[\frac{\sin\frac{\omega(x_i - x)}{c}}{\frac{\cos\omega l}{c}}\right] H_{n_i}. \tag{19}$$

The total wave function of the string system of oscillator is then

$$\Psi(x,t) = \Pi_i^{\infty} \psi_i(x,t) = \Pi_i^{\infty} \frac{Ac}{ES} N_{n_i} \left[\frac{\sin\frac{\omega(x_i - x)}{c}}{\frac{\cos\omega l}{c}}\right] H_{n_i}, \tag{20}$$

Or,

$$\Psi(x,t) = \frac{Ac}{ES}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \Pi_i^{\infty} \sqrt{\frac{1}{2^{n_i} n_i!}} \left[\frac{\sin\frac{\omega(x_i - x)}{c}}{\frac{\cos\omega l}{c}}\right] (-1)^{n_i} e^{\xi^2} \frac{d^{n_i}}{d\xi^{n_i}} \exp\left(-\xi^2/2\right). \tag{21}$$

with

$$\xi = (x_i - x)\sqrt{m\omega/\hbar}. \tag{22}$$

So, the quantization of string is possible only if we divide the string into elementary discrete points with supposing that in every point of string $X \in (0, l)$, there is a quantum oscillator with the stationary states described by eq. (19). There is an analogue representation to eq. (21), which was applied by Feynman for determination of the quantum theory of the Mössbauer effect (Feynman, 1972).

6.4 Discussion

The starting point for string theory is the idea that the point-like particles are modeled by one-dimensional objects called strings. Strings propagate through space and interact with each other. In a given version of string theory, there is only one kind of string, which may look like a small loop, or, segment of ordinary string, and it can vibrate in different ways. On distance scales larger than the string scale, a string will look just like an ordinary particle, with its mass, charge, and other properties determined by the vibrational state of the string. In this way, all of the different elementary particles may be viewed as vibrating strings. In string theory, one of the vibrational states of the string gives rise to the graviton, a quantum mechanical particle that carries gravitational force. Thus string theory is also theory of quantum gravity and replaces the quantum gravity with the gravitons with spin 2.

The string theory can be extended to the quark-quark interaction by the string potential, defined as the quark mass correction to the string potential, which was performed by Lambiase and Nesterenko (1996). The calculation of the interquark potential generated by a string with massive ends was performed by Nesterenko and Pirozhenko (1997), and others. The propagation of a pulse in the real strings and rods which can be applied to the two-quark system as pion and so on, was calculated by author (Pardy, 2005). So, it is not excluded that our aproach can be extended to generate the new way of the string theory of matter and space-time.

Let us only remark that author considered the string model of gravity where the gravitational mediation was modeled by string with the results, which are identical with the classical theory of gravity (Pars, 1964). The extra result was the vibration of a body at the end of the string, which it was still not confirmed by experiment (Pardy, 1996).

7 The string motion under the periodic local force

7.1 Introduction

We consider the string under local periodic force. We derive the quantum internal motion of this system. The quantization follows from the oscillator equations of quantum oscillator.

According to Nielsen and Olesen (1973) there is parallelism between the Higgs model of broken gauge invariance and the Landau-Ginzburg superconductivity theory on the one hand and the dual string model being the Abrikosov flux lines in superconductors II. So, dual string is mathematical realization of magnetic flux tube in equilibrium against the pressure of the surrounding charged superfluid. Only strings with no ends were considered by them (Nambu, 1974). The internal quantum motion of strings is not considered by the authors.

We consider here the string of the length l, the left and right ends of which are fixed and the string is under local periodic force $A\varrho \sin \omega t$. We derive the quantum internal motion of this system using the so called the oscillator quantization of the string.

The non-relativistic quantization of the equation for the energy of a free particle

$$\frac{p^2}{2m} = E \tag{1}$$

consists in replacing classical quantities by operators. We get the non-relativistic Schrödinger equation for a free particle. The operator replacings are $E \to i\hbar\partial/\partial t$, $\mathbf{p} \to -i\hbar\nabla$.

The Schrödinger equation suffers from not being relativistically covariant, meaning it does not take into account Einstein's special relativity.

It is natural to perform the special relativity generalization of the energy relation describing the energy:

$$E = \sqrt{p^2c^2 + m^2c^4}. \tag{2}$$

Then, just inserting the quantum mechanical operators for momentum and energy yields the equation

$$i\hbar\frac{\partial}{\partial t} = \sqrt{(-i\hbar\nabla)^2c^2 + m^2c^4}. \tag{3}$$

This, however, is a cumbersome expression to work with because the differential operator cannot be evaluated while under the square root sign.

Klein and Gordon instead began with the square of the above identity, i.e. $E^2 = p^2c^2 + m^2c^4$, which, when quantized, gives

$$\left(i\hbar\frac{\partial}{\partial t}\right)^2 = (-i\hbar\nabla)^2c^2 + m^2c^4. \tag{4}$$

So, we have seen that the quantization of classical mechanics is the simple replacing classical quantities by operators. We use here the novel quantization method where classical oscillators forming the classical systems are replaced simply by the quantum solution of quantum oscillators. The natural step is to apply the method to motion of the classical string.

7.2 The classical derivation of the string motion

The differential equation of motion of string elements can be derived by the following way (Tikhonov et al., 1977). We suppose that the force acting on the element dx of the string is given by the law:

$$T(x,t) = ES\left(\frac{\partial u}{\partial x}\right), \tag{5}$$

where E is the modulus of elasticity, S is the cross section of the string. We easily derive that

$$T(x+dx) - T(x) = ESu_{xx}dx. \tag{6}$$

The mass dm of the element dx is $\varrho ESdx$, where $\varrho = const$ is the mass density of the string matter and the dynamical equilibrium gives

$$\varrho S dx u_{tt} = ES u_{xx} dx. \tag{7}$$

So, we get

$$\frac{1}{a^2} u_{tt} - u_{xx} = 0; \quad a = \left(\frac{E}{\varrho}\right)^{1/2}. \tag{8}$$

Now, let us consider the following problem of the mathematical physics. The string is under the local periodic force $F = \varrho A \sin\omega t$ at point $0 < c < l$. With regard to the fact that we consider the additional periodical force $F = \varrho A \sin\omega t$ at point $0 < c < l$, we must reformulate the standard string problem of mathemtical physics as it follows. The motion of the string is u_1 in the interval $0 < x < c$ and u_2 in the interval $c < x < l$.

So, we solve the mathematical problem:

$$(u_1)_{tt} = a^2 (u_1)_{xx}; \quad 0 < x < c \tag{9a}$$

$$(u_2)_{tt} = a^2 (u_2)_{xx}; \quad c < x < l \tag{9b}$$

with the conditions

$$u_1(x = 0) = 0; \quad u_2(x = l) = 0 \tag{10a}$$

$$u_1(x = c) = u_2(x = c) \tag{10b}$$

$$E(u_1)_x(x = c) - E(u_2)_x(x = c) = \varrho A \sin\omega t. \tag{11}$$

We look for the solution in the form (Koshlyakov, et al., 1962)

$$u_1(x,t) = C_1 \sin\frac{\omega x}{a} \sin\omega t \tag{12a}$$

$$u_2(x,t) = C_2 \sin\frac{\omega(l-x)}{a} \sin\omega t. \tag{12b}$$

For the determination of the arbitrary constants C_1, C_2 with regard to the conditions $(10a), (10b), (11)$, we get the solution of our problem in the following form (Koshljakov, et al., 1962):

$$u_1(x,t) = \frac{A}{a\omega} \frac{\sin\frac{\omega(l-c)}{a}}{\sin\frac{\omega l}{a}} \sin\frac{\omega x}{a} \sin\omega t; \quad 0 < x < c \tag{13a}$$

$$u_2(x,t) = \frac{A}{a\omega} \frac{\sin\frac{\omega c}{a}}{\sin\frac{\omega l}{a}} \sin\frac{\omega(l-x)}{a} \sin\omega t; \quad c < x < l. \tag{13b}$$

So, we see that the string motion is a such that at every point $X \in (0, l)$ there is an oscillator with an amplitudes

$$A_1 = \frac{A}{a\omega} \frac{\sin \frac{\omega(l-c)}{a}}{\sin \frac{\omega l}{a}} \sin \frac{\omega x}{a}; \quad 0 < x < c \tag{14a}$$

$$A_2 = \frac{A}{a\omega} \frac{\sin \frac{\omega c}{a}}{\sin \frac{\omega l}{a}} \sin \frac{\omega(l-x)}{a}; \quad c < x < l. \tag{14b}$$

7.3 Quantization of the string motion by harmonic oscillators

It is well known that harmonic oscillator equation

$$\ddot{x} + \omega^2 x = 0; \quad \omega = \sqrt{k/m} \tag{15a}$$

has the solution

$$x(t) = A\cos(\omega t + \varphi). \tag{15b}$$

In case of the quantum mechanical oscillator motion, the solution for the stationary sates is (Grashin, 1974)

$$\psi_n = N_n H_n \exp\left(-\xi^2/2\right); \quad \xi = x\sqrt{m\omega/\hbar}, \tag{16}$$

where N_n is the normalization constant

$$N_n = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{1}{2^n n!}} \tag{17}$$

and H_n are the Hermite polynomials defined by the following relation

$$H_n = (-1)^n e^{\xi^2} \frac{d^n}{d\xi^n} \exp\left(-\xi^2/2\right). \tag{18}$$

So, the wave function of the one string oscillator of the string with the periodic force at point c in the form:

$$\psi_i(x,t) = \frac{A}{a\omega} \frac{\sin \frac{\omega(l-c)}{a}}{\sin \frac{\omega l}{a}} \sin \frac{\omega(x_i - x)}{a} N_{n_i} \frac{\cos \omega l}{c} H_{n_i}; \quad 0 < x < c \tag{19a}$$

and

$$\psi_i(x,t) = \frac{A}{a\omega} \frac{\sin \frac{\omega(l-c)}{a}}{\sin \frac{\omega l}{a}} \sin \frac{\omega(l - x_i + x)}{a} N_{n_i} \frac{\cos \omega l}{c} H_{n_i}; \quad c < x < l. \tag{19b}$$

The total wave function of the string system of oscillators is then

$$\Psi_1(x,t) = \Pi_i^\infty \psi_i(x,t) =$$

$$\Pi_i^\infty \frac{A}{a\omega}\frac{\sin\frac{\omega(l-c)}{a}}{\sin\frac{\omega l}{a}}\sin\frac{\omega(x_i-x)}{a}N_{n_i}\frac{\cos\omega l}{c}H_{n_i}; \quad 0<x<c \tag{20a}$$

and

$$\Psi_2(x,t) = \Pi_i^\infty \psi_i(x,t) =$$

$$\Pi_i^\infty \frac{A}{a\omega}\frac{\sin\frac{\omega(l-c)}{a}}{\sin\frac{\omega l}{a}}\sin\frac{\omega(l-x_i+x)}{a}N_{n_i}\frac{\cos\omega l}{c}H_{n_i}; \quad c<x<l. \tag{20b}$$

Or,

$$\Psi_1(x,t) = \frac{A}{a\omega}\frac{\sin\frac{\omega(l-c)}{a}}{\sin\frac{\omega l}{a}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

$$\Pi_i^\infty \sqrt{\frac{1}{2^{n_i}n_i!}}\frac{\cos\omega l}{c}(-1)^{n_i}e^{\xi^2}\frac{d^{n_i}}{d\xi^{n_i}}\exp\left(-\xi^2/2\right); \quad 0<x<c \tag{21a}$$

with

$$\xi = (x_i - x)\sqrt{m\omega/\hbar}. \tag{21$a\xi$}$$

and

$$\Psi_2(x,t) = \frac{A}{a\omega}\frac{\sin\frac{\omega(l-c)}{a}}{\sin\frac{\omega l}{a}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

$$\Pi_i^\infty \sqrt{\frac{1}{2^{n_i}n_i!}}\frac{\cos\omega l}{c}(-1)^{n_i}e^{\xi^2}\frac{d^{n_i}}{d\xi^{n_i}}\exp\left(-\xi^2/2\right); \quad c<x<l. \tag{21b}$$

with

$$\xi = (l - x_i + x)\sqrt{m\omega/\hbar}. \tag{21$b\xi$}$$

So, the quantization of string is possible only if we divide the string into elementary discrete points supposing that in every point of string $X \in (0,l)$, there is a quantum oscillator with the stationary states described by eq. (19). There is an analogue representation to eq. (21), which was applied by Feynman for determination of the quantum theory of the Mössbauer effect (Feynman, 1972).

7.4 Discussion

The starting point for string theory is the idea that the point-like particles are modeled by one-dimensional objects called strings. Strings propagate through space and interact with each other. In a given version of string theory, there is only one kind of string, which may look like a small loop, or, segment of ordinary string, and it can vibrate in different ways. On distance scales larger than the string scale, a string will look just like an ordinary particle, with its mass, charge, and other properties determined by the vibrational state of the string. In this way, all of the different elementary particles may be viewed as vibrating strings. In string theory, one of the vibrational states of the string gives rise to the graviton, a quantum mechanical particle that carries gravitational force. Thus string theory is also theory of quantum gravity and replaces the quantum gravity with the gravitons with spin 2.

The string theory can be extended to the quark-quark interaction by the string potential, defined as the quark mass correction to the string potential, which was performed by Lambiase and Nesterenko (1996). The calculation of the interquark potential generated by a string with massive ends was performed by Nesterenko and Pirozhenko (1997), and others. The propagation of a pulse in the real strings and rods which can be applied to the two-quark system as pion and so on, was calculated by author (Pardy, 2005). So, it is not excluded that our oscillator quantization of the string can be extended to generate the new way of the string theory of matter and space-time.

8 The quantization of the string motion with the interstitial massive defect

8.1 Introduction

We will consider the string, the left end of which is fixed at the beginning of the coordinate system, the right end is fixed at point l and mass m is fixed between the ends of the string. We determine the classical and the quantum vibration of such system. The proposed model can be also related in the modified form to the problem of the Mössbauer effect, or recoilless nuclear resonance fluorescence, which is the resonant and recoil-free emission and absorption of gamma radiation by atomic nuclei bound in a solid. (Mössbauer, 1958)

First, let us consider the string, the left end of which is fixed at the beginning of the coordinate system, the right end is fixed at point l and mass m is fixed interstitially between the ends of the string. The vibration motion of the string and the massive point with mass m is the problem of the mathematical physics in case that the tension is linearly dependent on elongation.

The differential equation of motion of string elements can be derived by the well known way. We suppose as in the part 1 that the string tension force acting on the element dl of the string is given by the law (Tikhonov et al., 1977):

$$T(x,t) = ES\left(\frac{\partial u}{\partial x}\right), \tag{1}$$

where E is the modulus of elasticity, S is the cross section of the string. We easily derive that

$$T(x+dx)-T(x)=ES\left(\frac{\partial u}{\partial x}\right)(x+dx)-ES\left(\frac{\partial u}{\partial x}\right)(x)\sim ESu_{xx}dx. \tag{2}$$

The mass dm of the element dl is ϱSdx, where ϱ is the mass density of the string matter and the dynamical equilibrium gives

$$\varrho Sdxu_{tt}=ESu_{xx}dx. \tag{3}$$

Or, after minimal modification we get the obligate result of the first part of our elaboration

$$\frac{1}{c^2}u_{tt}-u_{xx}=0; \quad c=\left(\frac{E}{\varrho}\right)^{1/2}. \tag{4}$$

The last procedure was performed evidently in order to get the wave equation.

8.2 The classical string motion with the interstitial massive point

Now, let us consider the string with the point-like mass at coordinate s in the interval $(0,l)$. Then, the left part string motion of the string be $u_1(x,t)$ and the right side of the string motion is $u_2(x,t)$. The corresponding equation of motion of both part of the string are as follows (Koshlyakov, et al., 1962):

$$(u_1)_{tt}=c^2(u_1)_{xx}; \quad (0<x<s), \tag{5a}$$

$$(u_2)_{tt}=c^2(u_2)_{xx}; \quad (s<x<l). \tag{5b}$$

The boundary and interstitial conditions are $S=1$:

$$u_1(x=0)=0, \quad u_2(x=l)=0. \tag{6}$$

$$u_1(x=s)=u_2(x=s). \tag{7}$$

The dynamical equation involving interstitial point is with $E=\varrho c^2$:

$$\varrho c^2(u_1)_x(s)-\varrho c^2(u_2)_x(s)=m(u_1)_{tt}(s)-m(u_2)_{tt}(s). \tag{8}$$

Let us look for the solution of the last equation in the form

$$u_1(x,t)=C_1\sin\frac{\omega x}{c}\sin\omega t \tag{9}$$

$$u_2(x,t)=C_2\sin\frac{\omega(l-x)}{c}\sin\omega t. \tag{10}$$

We see that the suggested solution is in harmony with the boundary conditions:

$$u_1(x = 0) = 0, \quad u_2(x = l) = 0. \tag{11}$$

After insertion of $u_1(x, t), u_2(x, t)$ from (9-10) into eqs. (7-8), we get the system of equations

$$C_1 \sin \frac{\omega s}{c} = C_2 \sin \frac{\omega(l - s)}{c} \tag{12}$$

and

$$C_1 \varrho c\omega \cos \frac{\omega s}{c} - C_2 \varrho c\omega \cos \frac{\omega(l - s)}{c} =$$

$$C_1 m\omega^2 \sin \frac{\omega s}{c} - C_2 m\omega^2 \sin \frac{\omega(l - s)}{c}. \tag{13}$$

In order to get the regular solution, the determinant of the system must be zero. Or,

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 0, \tag{14}$$

where

$$A = \sin \frac{\omega s}{c} \tag{15}$$

$$B = -\sin \frac{\omega(l - s)}{c} \tag{16}$$

$$C = \varrho c\omega \cos \frac{\omega s}{c} - m\omega^2 sin \frac{\omega s}{c} \tag{17}$$

$$D = -\varrho c\omega \cos \frac{\omega(l - s)}{c} + m\omega^2 \sin \frac{\omega(l - s)}{c}. \tag{18}$$

It follows from eq. (14)

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = AD - BC = 0. \tag{19}$$

Or,

$$\sin \frac{\omega s}{c} \left[-\varrho c\omega + m\omega^2 \tan \frac{\omega(l - s)}{c} \right] +$$

$$\sin \frac{\omega(l - s)}{c} \left[\varrho c\omega - m\omega^2 \tan \frac{\omega s}{c} \right] = 0, \tag{20}$$

So, we see, that the determination of the frequency ω involves the transcendent equation.The solution can be performed graphically, or by computer. Such problem is the integral part of the university mathematical methods (Arfken, 1967).

Nevertheless, it is evident that the trivial solution is for $\omega = 0$ and for

$$\omega s = \pi nc, \quad n = 0, 1, 2, ...; \quad \omega(l - s) = \pi kc, k = 0, 1, 2, ..., \tag{21}$$

which implies that the correspondence between l and k is only for

$$\frac{l}{s} = \frac{(k+n)}{n}. \tag{22}$$

The determination of the ω from the transcendent equations

$$\left[-\varrho c\omega + m\omega^2 \tan\frac{\omega(l-s)}{c}\right] = 0, \quad \left[\varrho c\omega - m\omega^2 \tan\frac{\omega s}{c}\right] = 0 \tag{23}$$

is difficult and it can be solved by the appropriate mathematical methods (Arfken, 1967).

8.3 The quantization of the string motion by harmonic oscillators

The non-relativistic quantization of the equation for the energy of a free particle

$$\frac{p^2}{2m} = E \tag{24}$$

consists in replacing classical quantities by operators. We get the non-relativistic Schrödinger equation for a free particle. The operator replacings are $E \to i\hbar\partial/\partial t$, $\mathbf{p} \to -i\hbar\nabla$.

The Schrödinger equation suffers from not being relativistically covariant, meaning it does not take into account Einstein's special relativity.

It is natural to perform the special relativity generalization of the energy relation describing the energy:

$$E = \sqrt{p^2c^2 + m^2c^4}. \tag{25}$$

Then, just inserting the quantum mechanical operators for momentum and energy yields the equation

$$i\hbar\frac{\partial}{\partial t} = \sqrt{(-i\hbar\nabla)^2c^2 + m^2c^4}. \tag{26}$$

This, however, is a cumbersome expression to work with because the differential operator cannot be evaluated while under the square root sign.

Klein and Gordon instead began with the square of the above identity, i.e. $E^2 = p^2c^2 + m^2c^4$, which, when quantized, gives

$$\left(i\hbar\frac{\partial}{\partial t}\right)^2 = (-i\hbar\nabla)^2c^2 + m^2c^4. \tag{27}$$

So, we have seen that the quantization of classical mechanics is the simple replacing classical quantities by operators. We use here the novel quantization method where classical oscillators forming the classical systems are replaced simply by the quantum solution of quantum oscillators. The natural step is to apply the method to motion of the classical string.

It is well known that harmonic oscillator equation

$$\ddot{x} + \omega^2 x = 0; \quad \omega = \sqrt{k/m} \tag{28a}$$

has the solution

$$x(t) = A\cos(\omega t + \varphi). \tag{28b}$$

In case of the quantum mechanical oscillator motion, the solution for the stationary sates is (Grashin, 1974)

$$\psi_n = N_n H_n \exp\left(-\xi^2/2\right); \quad \xi = x\sqrt{m\omega/\hbar}, \tag{29}$$

where N_n is the normalization constant

$$N_n = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{1}{2^n n!}} \tag{30}$$

and H_n are the Hermite polynomials defined by the following relation

$$H_n = (-1)^n e^{\xi^2}\frac{d^n}{d\xi^n}\exp\left(-\xi^2/2\right). \tag{31}$$

So, the wave function of the one string oscillator of the string with the periodic force at point c in the form:

$$\psi_i(x,t) = AC_1\sin\frac{\omega(x_i - x)}{c}N_{n_i}H_{n_i}; \quad 0 < x < c \tag{32a}$$

and

$$\psi_i(x,t) = AC_2\sin\frac{\omega(l - x_i + x)}{c}N_{n_i}H_{n_i}; \quad c < x < l. \tag{32b}$$

The total wave function of the string system of oscillators is then

$$\Psi_1(x,t) = \Pi_i^\infty \psi_i(x,t) =$$

$$\Pi_i^\infty AC_1\sin\frac{\omega(x_i - x)}{c}N_{n_i}H_{n_i}; \quad 0 < x < c \tag{33a}$$

and

$$\Psi_2(x,t) = \Pi_i^\infty \psi_i(x,t) =$$

$$\Pi_i^\infty AC_2\sin\frac{\omega(l - x_i + x)}{c}N_{n_i}H_{n_i}; \quad c < x < l. \tag{33b}$$

So, the quantization of string is possible only if we devide the string into elementary discrete points supposing that in every point of string $X \in (0, l)$, there is a quantum oscillator with the stationary states described by eq. (32). There is an analogue representation to eq. (21), which was applied by Feynman for determination of the quantum theory of the Mössbauer effect (Feynman, 1972).

8.4 Discussion

The articles was inspired by the author diploma work (Pardy, 1965), in which the interaction of light with the crystal defect was calculated. At this theory the crystal was replaced by the Euler-Bernoulli linear chain (Landau, et al., 1965) with some defects.

The proposed model can be also related in the modified form to the problem of the Mössbauer effect, or recoilless nuclear resonance fluorescence, which is the resonant and recoil-free emission and absorption of gamma radiation by atomic nuclei bound in a solid. (Mössbauer,1958). In this effect, a narrow resonance for nuclear gamma emission and absorption results from the momentum of recoil transited to a surrounding crystal lattice and not to the emitting or absorbing nucleus alone. No gamma energy is lost. Emission and absorption occur at the same energy, resulting in strong, resonant absorption.

The generalization of our continual model can be performed in such a way that we replace one massive point m by the massive points $m_1, m_2, m_3, ...\ \ m_k$ at points $s_1, s_2, s_3, ...\ \ s_k$ and solve the adequate system of differential equations.

The string theory can be extended to the quark-quark interaction by the string potential, defined as the quark mass correction to the string potential, which was performed by Lambiase and Nesterenko (1996). The calculation of the interquark potential generated by a string with massive ends was performed by Nesterenko and Pirozhenko (1997), and others. The propagation of a pulse in the real strings and rods which can be applied to the two-quark system as pion and so on, was calculated by author (Pardy, 2005). So, it is not excluded that our oscillator quantization of the string can be extended to generate the new way of the string theory of matter and space-time.

9 Appendix: Some remarks on the action at a distance

The starting point of the problem of the action at a distance is the question "How do bodies act on one another across space?". Or, what is the explanation of of action at a distance. There are the various answers to this question to illustrate the role of fundamental analogies or models in physics. We have seen that one model is the string model of interaction. This is the mechanical model which can be considered as the integral part of Leibniz philosophy. Leibniz writes: ... " all the phenomena of bodies can be explained mechanically, or by the corpuscular philosophy, according to certain principles of mechanics, which are laid down without taking into consideration whether there are souls or not". (Leibniz, 1686).

But from the viewpoint of phenomenology, Leibniz appeals to the principle of continuity to show that there must be continuity of cause and effect in space and time. Hence action involves contact. Or, "A body is never moved naturally, except by another body which touches it and pushes it; after that it continues until it is prevented by another body which touches it. Any other kind of operation on bodies is either miraculous or imaginary. " Attraction can only happen in an explicable manner, i.e. by an impulsion of subtler bodies. So, we cannot admit that attraction is a primitive quality essential to matter (Hesse, 2005).

According to Nielsen and Olesen (1973) there is parallelism between the Higgs model of broken gauge invariance and the Landau-Ginzburg superconductivity theory on the one hand and the dual string model being the Abrikosov vortex lines in superconductors II. So, dual string is mathematical realization of magnetic vortex tube in equilibrium against the pressure of the surrounding charged superliquid. Only strings with no ends were considered by them (Nambu, 1974). The internal quantum motion of strings is not considered by the authors.

The proposed model with the string with massive ends can be also related in the modified form to the recent problems of the radial motion of quarks bound by a strings, so called quark-string model of mesons and used to calculate the excited states of such system. The vibration energy of such states are involved in formula (42). The quantum version is necessary to elaborate. The recent analysis of such problem was performed by Lambiase and Nesterenko (1996) and Nesterenko and Pirozhenko (1997), and others. We hope that our approach and their approach will be unified to generate the new way of the string theory of matter and space-time.

The determination of the hadron mass spectrum in the framework of quantum chromodynamics (QCD), or dynamical states of mesons still remains a unsolved problem. So, the potential methods and the string methods are used for this purpose. In the string model of hadrons the quarks are treated to be tied together by a gluon tube which can be approximated by the tube of vanishing width, or by string (Nambu, 1974). Then, dynamics of hadrons can be approximated by the Nambu-Goto action for the relativistic string. The merit of the string model is the natural explanation of the quark confinement and the dynamics of the system. On the other hand there are some mathematical problems when the quark masses are different from zero. So, there are many trials to give the final words to the problem of hadron dynamics and hadron masses. It is well known that some models of string theory involves also the so called extra-dimension. However, it is well known that this theory is at present time not predictable. The goal of theoretical physics is predictions based on the mathematical knowledge. On the other hand the goal of the experimental physic is the confirmation of theory with appropriate experimental simplicity or virtuosity.

The starting point of the string approach to the two-body problem with massive quarks at the ends of string is based on the action (1)-section **5**. From this action the equation of motion for the rotating string can be derived and it is possible to show that this model is integrable.

References

Arfken, G. *Mathematical Methods for Physicists* (Academic Press, New York and London, 1967).

Bardeen, W. Bars, A. I., Hanson, A. J. and Barbashov B. M. and Nesterenko, V. V. (1990). *Introduction to the relativistic string theory*, Singapore, World Scientific.

Bell, J. S., *Speakable and Unspeakable in Quantum Mechanics* (Cam- bridge: Cambridge University Press, 1993) , pp 67-68

Bentley, R. *Works of Richard Bentley*, Vol. 3, pp. 210-211,; Letter of Newton to Bentley, Trinity College, Jan 17, 1692-1693.

Braginsky, V. B., Caves, C. M. and Thorne, K. S. (1977). Laboratory experiments to test relativistic gravity, Phys. Rev. D **15**, 2047.

Einstein, A. (1911). Über den Einflus der Schwerkraft auf die Ausbreitung des Lichtes, Ann. Phys. 1911, **35**, 898-908. ibid. *On the Influence of Gravitation on the Propagation of Light*, Vol. 3, Doc. 23, 485-497 (Einstein 1911h).

Einstein, A. (1917). Kosmologische Betrachtungen zur allgemeinen Relativitaetstheorie, Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften Berlin. part 1: 142-152.

Einstein, A. (1918). Principieless zur allgemeinen Relativitätstheorie, Ann. Phys. 1918, **55**, 241-244. ibid. *On the Foundations of the General Theory of Relativity* Vol. 7, Doc. 4, 37-44 (Einstein 1918e).

Feynman, R., Morinigo, F. B. and Wagner, W. B. *Feynman Lectures on Gravitation*, (Pinguin Books, London, 1995).

Feynman, R. P. *Statistical mechanics*, (W. A. Benjamin, Inc., Reading, Massachusets, 1972).

Fock, V. *Theory of space, time and and gravitation*, Second revised ed., (Pergamon Press, Oxford, London, ... 1964).

Frampton, P. (1975). String approaches to hadron structure, Phys. Rev. D **12**, 538.

Franklin, J. (2010). Lorentz contraction, Bell's spaceships, and rigid body motion in special relativity, arXiv:0906.1919v3.

Fraser, G. E., Lillestøl, E. and Sellevåg, I. (1998). *The Search for Infinity*, George Philip

Limited.

Frova, A. and Marenzana, M. *Thus Spoke Galileo, The Great Scientist's Ideas and Their Relevance to the Present Day*, (Oxford University Press, 2006).

Grashin, A. F. (1974). *Quantum mechanics*, (Enlightenment, Moscow). (in Russian).

Halliday, D. and Resnick, R. *PHYSICS, I II* (John Wiley and Sons; New York, London, Sydney) (1966).

Hesse, M. B., (2005). *Forces and Fields The Concept of Action at a Distance in the History of Physics*, (DOVER PUBLICATIONS, INC. Mineola, New York, unabridged republication of the first edition originally published in London and New York by T. Nelson in 1961.)

Chandrasekhar, S. (1972). On the "Derivation" of Einstein's Field Equations, AJP **40**, No. 2, pp. 224 - 234.

Chodos, A. and Thorn, C. B. (1974). Making the massless string massive, Nucl. Phys. **72**, 509.

Kenyon, I. R. *General Relativity*, (Oxford University Press Inc. New York, 2-nd ed., 1996).

Koshlyakov, N. S., Gliner, E. V. and Smirnov, M. M. *The Differential Equations of Mathematical Physics.* (Nauka, Moscow, 1962). (in Russsian).

Lambiase, G. and Nesterenko, V. V. (1996). Quark mass correction to the string potential. *Phys. Rev. D* **54**, 6387.

Landau, L. D. and Lifshitz, E. M. , *The Classical Theory of Fields*, Fourth revised English edition, (Amsterdam, Boston, Heidelberg, London, ..., 1987).

Landau, L. D. and Lifschitz, E. M. *Mechanics* (Nauka, Moscow, 1965). (in Russian).

Lebedev, N. N., Skalskaya, I. P. and Uflyand, Ya. S. *Collection of Problems of Mathematical Physics*, (GITTL, Moscow, 1955). (in Russsian).

Leibniz, G. W. (1686). Letter to Arnaud (1686), *Die Philosophischen Schriften von G. W. Leibniz*, ed. Gerhardt, Berlin, 1875-90, p. 78.

Lyle, S. N. *Uniformly Accelerating Charged Particles, A Threat to the Equivalence Principle*, (Springer-Verlag, Berlin, Heidelberg, 2008).

Mössbauer, R. L. (1958).Kernresonanzfluoreszenz von Gammastrahlung in Ir191. Zeitschrift für Physik A (in German). **151** (2) 124-143

Nambu Y. (1974). Strings, Monopoles and gauge fields. *Physical Review D* **10**, 4262.

Newton, I. *Principia Mathematica* (University of California Press, Berkeley and Loss Angeles, 1966). p. 634.

Nesterenko, V. V. (1990). On the radial motion of quarks bound by a string, Z. Phys. C - Particles and Fields **47**, 111.

Nesterenko, V. V. and Pirozhenko, I. G. (1997). Calculation of the interquark potential generated by a string with massive ends. *Phys. Rev. D* **55**, 6603.

Nielsen, H. B. and Olesen, P. (1973). Vortex-line models for dual strings, Nucl. Phys. **B61**, 45.

Pardy, M. (1969). A remark on the clock paradox, Phys. Lett. 28 A, No. 11, 766.

Pardy, M. (1980). The string model of action at a distance, University of J. E. Purkynje (UJEP), Unpublished.

Pardy, M. (1983). The synchrotron production of gravitons by the binary system, GRG **15**, No. 11, 1027.

Pardy, M. (1994a). The High-energy gravitons from the binary: Preprint: CERN-TH. 7239/94.

Pardy, M. (1994b). The quantum energy loss of a binary system: CERN-TH. 7299/94.

Pardy, M. (1994c). The gravitational Čerenkov radiation with radiative corrections: CERN-TH.7270/94.

Pardy, M. (1994d). The gravitational Čerenkov radiation with radiative corrections: Phys. Lett. B **336**, 362.

Pardy, M. (1996). The string model of gravity, arXiv:gr-qc/9602007v1, 5 Feb.

Pardy, M. (1997). The Čerenkov effect and the Lorentz contraction, Phys. Rev. A 55 (3) (1997) 1647-1652,

Pardy, M. (2002). Electron in the ultrashort laser pulse. e-print hep-ph/0207274.

Pardy, M. (2003). Electron in the ultrashort laser pulse. *Int. Journal of Theoretical Physics*, **41**, No. 1, 99.

Pardy, M. (2004). Light in metric space-time and its deflection by the screw dislocation. e-print hep-ph/0402249; ibid., *Spacetime Physics Research Trends. Horizons in World Physics*, Volume **248**, ISBN 1-59454-322-4 (2005).

Pardy, M. (2005). The propagation of a pulse in the real strings and rods, e-print math-ph/0503003.

Pardy, M. (2008). The relativistic length contraction of the carbon dumbbell moving in the LHC, or, ILC, arXiv: 0710.3489v2.

Pardy, M. (2009), The bremsstrahlung equation for the spin motion in electromagnetic field, *International Journal of Theoretical Physics*, Volume 48, Number 11, pp. 3241-3248; arXiv: hep-ph/0809.3888 v1, acc-ph/0801.2483.

Pardy, M. (2011). The radiation of the gravitational and electromagnetic binary pulsars; pp. 97-123. in: *Pulsars: Discoveries, Functions and Formation* (Editors: Peter A. Travelle, Nova Science Publishers, Series: Space Science).

Pardy, (2016). The excited mesons due to laser pulse, Intellectual Archive, 2016-09-26, ID 1764.

Pardy, M. (2018). The black hole binary gravitons and thermodynamics, viXra: 1810.0304v1.

Pardy, M. (2019). The Black Hole Binary Gravitons and Related Problems, IntechOpen. Book title: *New Ideas Concerning Black Holes and the Evolution of the Universe*

Pardy, M. (2020). The Zitterbewegung of Planets and Moon in the String Gravity, Intell. Arch., Vol. 9, No. 2, April-June, pp. 10-17.

Pars, L. A. *A Treatise on Analytical Mechanics*, (Heineman, London, 1964).

Petkov, V. (2009). Accelerating spaceships paradox and physical meaning of length contraction, arXiv:0903.5128v1.

Rindler, W. *Relativity, Special, General and Cosmological*, (Oxford University Press, 2003).

Schwinger, J. *Particles, Sources and Fields*, (Addison-Wesley Publishing Company, Reading, Mass., 1970).

Stanford Encyclopedia of Philosophy, (2006), Newton's Philosophy.

Tikhonov, A. N. and Samarskii, A. A. *The Equations of Mathematical Physics.* (Nauka, Moscow, 1977). (in Russsian).